Maltsev conditions for general congruence meet-semidistributive algebras

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Abstract

Meet semidistributive varieties are in a sense the last of the most important classes in universal algebra for which it is unknown whether it can be characterized by a strong Maltsev condition. We present a new, relatively simple Maltsev condition characterizing the meet-semidistributive varieties, and provide a candidate for a strong Maltsev condition.

1 Introduction

The tame congruence theory (TCT) [4], a structure theory of general finite algebras, has revealed that there are only 5 possibly local behaviors of a finite algebra:

(1) algebra having only unary functions,

(2) one-dimensional vector space,

(3) the two-element boolean algebra,

(4) the two-element lattice,

(5) the two element semilattice.

If there is a local behavior of type (i) in an algebra $\mathbf{A}$, the algebra is said to *have* type (i). A $\mathcal{V}$ variety have type (i) if there is an algebra $\mathbf{A} \in \mathcal{V}$ that have (i). If an algebra or variety does not have a type (i), it is said to omit type (i). The set of "bad" types that are omitted in a variety is an important structural information; for instance, it plays a significant role in the fixed-template constraint satisfaction problem [3]. The "worst" type is type (1) and omitting it has been characterized in many equivalent ways, one of which is given in the following theorem.

Theorem 1.1. *[9] A locally finite variety $\mathcal{V}$ omits type (1) if and only if there is an idempotent WNU (weak near unanimity) term in $\mathbf{A}$, that is a term satisfying the following identities:*

- *idempotence: $t(x, x, x, \ldots, x) = x$,*
- *weak near unanimity: $t(y, x, x, \ldots, x) = t(x, y, x, \ldots, x) = \cdots = t(x, \ldots, x, y)$*

for any $x, y \in \mathbf{A}$.

Such a characterization of varieties of algebras by means of the existence of terms satisfying certain identities are in general called Maltsev conditions. More precisely, a strong Maltsev condition is given by a finite set of term symbols and a finite set of identities. A given strong Maltsev condition is satisfied in a variety $\mathcal{V}$ if we can substitute the term symbols by actual terms in the variety in such a way that all the identities are satisfied. A general Maltsev condition is then a disjuction of countably many strong Maltsev conditions (as in the example of Theorem 1.1).

Whenever a variety $\mathcal{V}$ satisfies a certain Maltsev condition and $\mathcal{W}$ is interpretable into $\mathcal{W}$, then $\mathcal{W}$ satisfies the Maltsev condition too. For the notion of interpretability, we refer the reader to [4]. There are following relations between types of locally finite varieties and the interpretability.

- Any variety that has type (1) is interpretable into any variety.
- Any variety is interpretable into a variety that has type (3).
- Any variety that has type (5) is interpretable into a variety that has type (4).

Therefore, it is reasonable to ask for the Maltsev conditions for the following classes:

$$\mathcal{M}_{\{1\}}, \mathcal{M}_{\{1,2\}}, \mathcal{M}_{\{1,5\}}, \mathcal{M}_{\{1,2,5\}}, \mathcal{M}_{\{1,4,5\}}, \mathcal{M}_{\{1,2,4,5\}},$$

where $\mathcal{M}_S$ is the class of all the algebras that omits all the types from the set S. There is an appropriate Maltsev condition for all six classes.

It was proved that $\mathcal{M}_{\{1\}}$ and $M_{\{1,2\}}$ can be characterized by strong Maltsev conditions. Recall that idempotent term is a term t satisfying the equation $t(x, x, \ldots, x) = x$.

Theorem 1.2. *[7] A locally finite variety omits type (1) if and only if it has an idempotent 4-ary term s satisfying $s(r, a, r, e) = s(a, r, e, a)$.*

Theorem 1.3 (Theorem 2.8 of [8])**.** *A locally finite variety omits types (1) and (2) if and only if it has three-ary and four-ary idempotent terms w_3, w_4 satisfying equations*

$$w_3(yxx) = w_3(xyx) = w_3(xxy) = w_4(yxxx)$$

$$= w_4(xyxx) = w_4(xxyx) = w_4(xxxy).$$

In the same paper [8] the authors have demostrated that the remaining classes, that is $\mathcal{M}_{1,5}, \mathcal{M}_{1,2,5}, \mathcal{M}_{1,4,5}, \mathcal{M}_{1,2,4,5}$, cannot be characterized by strong Maltsev conditions.

Although types in the TCT are defined only for locally finite varieties (because only finite algebras are assigned types), the type-omitting classes have alternative characterizations which do not refer to the type-set. They are shown in the following table taken from [8].

Type Omitting Class	Equivalent property,
$\mathcal{M}_{\{1\}}$	satisfies a nontrivial idempotent Maltsev condition,
$\mathcal{M}_{\{1,5\}}$	satisfies a nontrivial congruence identity,
$\mathcal{M}_{\{1,4,5\}}$	congruence n-permutable, for some $n > 1$,
$\mathcal{M}_{\{1,2\}}$	congruence meet semidistributive,
$\mathcal{M}_{\{1,2,5\}}$	congruence join semidistributive,
$\mathcal{M}_{\{1,2,4,5\}}$	congruence n-permutable for some n and congruence join semidistributive.

Each of the properties in the right column of the table is characterized by an idempotent Maltsev condition [4] for general (not necessarily locally finite) varieties. However, Theorems 1.2, 1.3 giving strong Maltsev conditions are not guaranteed to work. Indeed, there is an example of an idempotent algebra that satisfy a non-trivial Maltsev condition, but has no term $s(r,a,r,e) = s(a,r,e,a)$, see [5]. However, it turned out that the first property is characterized by another strong Maltsev condition.

Theorem 1.4. *[11] An idempotent algebra satisfy a non-trivial Maltsev condition if and only if it has a term t such that*

$$t(yxx, xyy) = t(xyx, yxy) = t(xxy, yyx).$$

The finite counterexamples to strong Maltsev conditions for

$$\mathcal{M}_{\{1,5\}}, \mathcal{M}_{\{1,2,5\}}, \mathcal{M}_{\{1,4,5\}}, \mathcal{M}_{\{1,2,4,5\}}$$

work as counterexamples for the general case, so the remaining question is the following.

Question 1.1. *Is there a strong Maltsev condition that is equivalent to congruence meet-semidistributivity?*

1.1 Congruence meet-semidistributivity

By $\mathbf{Con}(\mathbf{A})$ we denote the lattice of congruences of $\mathbf{A}$. A variety $\mathcal{V}$ is said to be congruence meet-semidistributive (shortly $SD(\wedge)$) if for any $\mathbf{A} \in \mathcal{V}$, and any three congruences $\alpha, \beta, \gamma \in \mathbf{Con}(\mathbf{A})$ such that

$$\alpha \wedge \gamma = \beta \wedge \gamma,$$

we have

$$\alpha \wedge \gamma = \beta \wedge \gamma = (\alpha \vee \beta) \wedge \gamma.$$

This property has many equivalent definitions, see Theorem 8.1 in [2], we mention some of them.

Theorem 1.5. *Let $\mathcal{V}$ be a variety. The following are equivalent.*

- *$\mathcal{V}$ is a congruence meet-semidistributive variety.*
- *No member of $\mathcal{V}$ has a non-trivial abelian congruence.*
- *$[\alpha, \beta] = \alpha \wedge \beta$ for all $\alpha, \beta \in \mathbf{Con}(\mathbf{A})$ and all $\mathbf{A} \in \mathcal{V}$, where $[\alpha, \beta]$ denotes the commutator of congruences.*
- *The diamond lattice M_3 is not embeddable in $\mathbf{Con}(\mathbf{A})$ for any $\mathbf{A} \in \mathcal{V}$,*
- *$\mathcal{V}$ satisfies an idempotent Maltsev condition that fails in any finite one-dimensional vector space over a non-trivial field (equivalently in any module).*

In this paper we are going to study the Maltsev conditions satisfied by every $SD(\wedge)$ variety. Not only is it not known whether there is a strong Maltsev condition characterizing the $SD(\wedge)$ varieties, but the known Maltsev conditions for $SD(\wedge)$ were quite complicated. Probably the simplest Maltsev condition for $SD(\wedge)$ which was available before this work is the following one.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Consider some n, and a self-inverse bijection $\varphi\colon [2n] \to [2n]$ without fixed points, such that whenever $i < j < \varphi(i)$, then also $i < \varphi(j) < \varphi(i)$. Such a bijection corresponds to a proper bracketing sequence with n opening and n closing brackets. Then the *bracket terms* are ternary terms $b_1, \ldots, b_{2n}$ satisfying the following identities

$$b_1(x, y, z) = x, \quad b_{2n}(x, y, z) = z,$$

$$b_{2i}(y, x, x) = b_{2i-1}(y, x, x), \quad b_{2i}(x, x, y) = b_{2i+1}(x, x, y),$$
$$b_i(x, y, x) = b_{\varphi(i)}(x, y, x),$$

for any i where it makes sense.

Theorem 1.6 (Theorem 1 in [1])**.** *A variety $\mathcal{V}$ satisfies the $SD(\wedge)$ property if and only if it has some bracket terms.*

1.2 The new terms

In this paper we define (m_1+m_2)-terms as a triple of idempotent terms (f, g_1, g_2), where g_1 is m_1-ary, g_2 is m_2-ary, f is (m_1+m_2)-ary, and they satisfy the identities

$$f(x, x, \ldots, x, \underset{i}{y}, x, \ldots, x) = g_1(x, x, \ldots, x, \underset{i}{y}, x, \ldots, x) \text{ for any } i = 1, \ldots, m_1,$$

$$f(x, x, \ldots, x, \underset{n_1+i}{y}, x, \ldots, x) = g_2(x, x, \ldots, x, \underset{i}{y}, x, \ldots, x) \text{ for any } i = 1, \ldots, m_2.$$

We prove the following theorem.

Theorem 1.7. *A variety $\mathcal{V}$ is congruence meet-semidistributive if and only if it has $(3+m)$-terms for some m.*

Checking the backward implication is easy. For a contradiction, assume that the identities of (m_1+m_2)-terms were satisfied in modules. That means that f, g_1, g_2 are represented by linear combinations. In particular, let

$$f(x_1, x_2, \ldots, x_{m_1+m_2}) = a_1x_1 + \cdot + a_{m_1+m_2}x_{m_1+m_2}$$

$$g_1(x_1, x_2, \ldots, x_{m_1}) = b_1x_1 + \cdot + b_{m_1}x_{m_1}, \quad g_2(x_1, x_2, \ldots, x_{m_2}) = c_1x_1 + \cdot + c_{m_2}x_{m_2},$$

By plugging $x = 0, y \neq 0$ into the identities for f and g_1, we get $a_i = b_i$ for $i = 1, \ldots, m_1$. If we make the same substitution in the second identity, we get $a_{m_1+i} = c_i$ for $i = 1, \ldots, m_2$. Moreover, idempotency identity enforces

$$\sum_{i=1}^{m_1+m_2} a_i = \sum_{i=1}^{m_1} b_i = \sum_{i=1}^{m_2} c_i = 1.$$

Therefore we get

$$1 = \sum_{i=1}^{m_1+m_2} a_i = \sum_{i=1}^{m_1} b_i + \sum_{i=1}^{m_2} c_i = 2,$$

which contradicts that our field was non-trivial. Thus, we proved the backward implication.

To prove the forward implication, we take a detour through a generalized version of (m_1+m_2)-terms. Given n, m, we define $n \times (n+1) \times m$-terms as follows.

Let i have values from 1 to n, j have values from 1 to $n+1$, and k have values from 1 to m. The $n \times (n+1) \times m$-terms are idempotent $(n+1)m$-ary terms f_i (variables are indexed by pairs (j, k)) and idempotent nm-ary terms g_j (variables are indexed by pairs (i, k)) such that for every i, j, k they satisfy the equation

$$f_i(x, x, \ldots, x, \underset{(j,k)}{y}, x, \ldots, x) = g_j(x, x, \ldots, x, \underset{(i,k)}{y}, x, \ldots, x).$$

By definition, $1 \times 2 \times m$-terms are equivalent to the $(m+m)$-terms. On the other hand, for large enough n, m, it is simple to derive the $n \times (n+1) \times m$-terms from another Maltsev condition not satisfiable in vector spaces.

Proposition 1.1. *Let $\mathcal{V}$ be a $SD(\wedge)$ variety. Then $\mathcal{V}$ has $n \times (n+1) \times m$-terms for some n, m.*

Proof. By Theorem 1.6, we may assume that there are bracket terms $b_1, \ldots, b_{2n}$ corresponding to a bijection $\varphi\colon [2n] \to [2n]$. Notice that since φ forms a proper bracketing, $\varphi(i)$ has a different parity than i for any i. Let $\psi(i) = \varphi(2i-1)/2$ and $\psi'(i) = (\varphi(2i)+1)/2$. In other words, we splited $[2n]$ to odd and even part and labeled them as $[n]$; then ψ corresponds to the mapping φ odd $\to$ even, and ψ' to its inverse. We construct $n \times (n+1) \times 3$-terms as follows. We set

$$\begin{aligned}
g_1(x_{1,1}, \ldots, x_{\varphi(1),2}, \ldots) &= x_{1,1} = b_1(x_{1,1}, x_{\psi(1),2}, x), \\
g_i(\ldots, x_{i,1}, \ldots, x_{\psi(i),2}, \ldots, x_{i-1,3}, \ldots) &= b_{2i-1}(x_{i,1}, x_{\psi(i),2}, x_{i-1,3}) \\
g_{n+1}(\ldots, x_{n,3}) &= x_{n,3} \\
f_i(\ldots, x_{i,1}, \ldots, x_{\psi'(i),2}, \ldots, x_{i+1,3}, \ldots) &= b_{2i}(x_{i,1}, x_{\psi'(i),2}, x_{i+1,3})
\end{aligned}$$

All the $n \times (n+1) \times 3$-identities follows directly from the bracket identities. □

1.3 Outline

The rest of the proof is divided into two sections. In Section 2 we show that in $n \times (n+1) \times m$-terms, we can decrease n by one increasing m enough. It follows that any $SD(\wedge)$ variety has $(m+m)$-terms a large enough m. In Section 3, we improve that result to $(3+m)$-terms. Section 4 then provides a few counterexamples showing that requesting $(2+m)$-terms would be too strong. Finally, in Section 5 we discuss remaining open questions.

2 Simplifying $n \times (n+1) \times m$-terms

2.1 Semirings

We will need some basic facts about semirings for our first proof.

Semiring is a general algebra $\mathbf{A} = (A, +, \cdot, 0, 1)$ where $(A, +, 0)$ is a commutative monoid, $(A, \cdot, 1)$ is a monoid, zero absorbes everything in multiplication $(0 \cdot x = x \cdot 0 = 0)$, and distributive laws are satisfied, that is, $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(a+b) \cdot c = a \cdot c + b \cdot c$. As usual, the binary multiplication operation $\cdot$ is often ommited writing ab instead of $a \cdot b$.

Let $\mathcal{A}$ be an alphabet. The elements of the free monoid $\mathcal{A}^*$ generated by $\mathcal{A}$ are represented by finite words in the alphabet, multiplication concatenates the words and the constant 1 corresponds to the empty word. Finally, the elements of the free semiring generated by $\mathcal{A}$ are represented as finite multisets (formal sums) of words in $\mathcal{A}^*$. The addition in the free semiring is defined as sums (disjoint unions) of the corresponding multisets, and the product $p \cdot q$ is defined as piecewise product of the monomials, that is $\{u \cdot v : u \in p, v \in q\}$.

Let $\mathbf{F}$ be the free semiring generated by some alphabet $\mathcal{A}$, and E be a set of equations of the form $e_1 = 1, e_2 = 1, e_3 = 1, \ldots$ where $e_i \in \mathbf{F}$. We are going to provide a description of the conguence on $\mathbf{F}$ generated by E.

Take a monomial $u \in \mathcal{A}^*$. By a single expansion of u we mean any element of $\mathbf{F}$ of the form ve_iw where $vw = u$. A single expansion on a general element of $\mathbf{F}$ is then defined as performing a single expansion on one of its summands. Finally, we say that p is an expansion of q if we can obtain p by performing consecutive single expansion steps on q.

Proposition 2.1. *For any pair (p, q) of elements in $\mathbf{F}$, these two elements are congruent modulo the congruence generated by E if and only if there is a common expansion r of both p and q.*

Proof. The backward implication is obvious: If r is an expansion of p, then r is clearly congruent to p. Analogously, r is congruent to q, therefore p is congruent to q. We are going to prove the forward implication.

For $p, q \in \mathbf{F}$ we define a relation $p \sim q$ if there is a common expansion of p and q. Clearly each $e_i \sim 1$. To show that $\sim$ includes the congruence generated by E, it remains to prove that $\sim$ is a congruence. Symmetry and reflexivity is apparently satisfied, so we have to prove that $\sim$ is transitive and compatible with the operations. To do that, let us introduce some notation.

Let $p \leq q$ denote that q is an expansion of p and let $p \preccurlyeq q$ denote that q can be obtained by applying single expansion steps on a subset of summands of p. So $p \preccurlyeq q$ is stronger than $p \leq q$ but weaker that q being a single expansion of p.

These orderings are clearly closed under addition. In particular, if $p = \sum_i^n p_i$, $q = \sum_i^n q_i$ and $p_i \preccurlyeq q_i$, then $p \preccurlyeq q$.

Claim 2.1. *For any $p, q, r, s \in \mathbf{F}$ such that $p \preccurlyeq q$ we have $rps \preccurlyeq rqs$.*

To verify that, let $p = \sum_i^P p_i$, $q = \sum_i^P q_i$, $r = \sum_i^R r_i$, $s = \sum_i^S s_i$, where p_i, r_i, s_i are monomials and $p_i \preccurlyeq q_i$. Then

$$rps = \sum_i^R \sum_j^P \sum_k^S r_i p_j k_k, \quad rqs = \sum_i^R \sum_j^P \sum_k^S r_i q_j s_k.$$

Since $p_j \preccurlyeq q_j$, we can write $p_j = u_j v_j$ so that $q_j = u_j x_j v_j$ where $x_j \succcurlyeq 1$, that is, $x = 1$ or x one of the elements e_i. So we can write $r_i p_j s_k = (r_i u_j)(v_j s_k)$ and $r_i q_j = (r_i u_j) x_j (v_j s_k)$. Therefore $r_i p_j s_k \preccurlyeq r_i q_j s_k$ and thus $rps \preccurlyeq rqs$.

Claim 2.2. *For any $p, q, r \in \mathbf{F}$ such that $r \preccurlyeq p$ and $r \preccurlyeq q$ there exists $s \in \mathbf{F}$ such that $p \preccurlyeq s$ and $q \preccurlyeq s$.*

First, we prove the claim if r is a monomial. So polynomials p, q are constructed by inserting p', q' somewhere into r respectively, where $p', q' \succcurlyeq 1$. Without loss of generality, q' is inserted at the same position as p' or later, so we can write $r = uvw$, $p = up'vw$, $q = uvq'w$. Now we choose $s = up'vq'w$. By Claim 2.1 and $p', q' \succcurlyeq 1$ we get the required

$$p = (up'v)(w) \preccurlyeq (up'v)q'(w) = s, \quad q = (u)(vq'w) \preccurlyeq (u)p'(vq'w) = s.$$

For a general $r = \sum_i^n r_i$ where r_i are monomials, we decompose $p = \sum_i^n p_i$, $q = \sum_i^n q_i$ so that $r_i \preccurlyeq p_i, q_i$. Therefore, we find elements s_i such that $s_i \succcurlyeq p_i, q_i$, and eventually $s = \sum_i^n \succcurlyeq p, q$.

We are finally ready to prove the transitivity of $\sim$ and compatibility with operations.

Claim 2.3. *If $x, r, y \in \mathbf{F}$, $x \sim r$ and $r \sim y$, then $x \sim y$.*

By definition of $\sim$, there are $p, q \in \mathbf{F}$ such that $x, r \leq p$ and $r, y \leq q$. We break the expansion $r \leq p$ into finite number of single expansion steps getting a sequence

$$r = s_{0,0} \preccurlyeq s_{1,0} \preccurlyeq \cdots \preccurlyeq s_{P,0} = p.$$

Similarly, there is a sequence

$$r = s_{0,0} \preccurlyeq s_{0,1} \preccurlyeq \cdots \preccurlyeq s_{0,Q} = q.$$

By repeated application of Claim 2.2, we fill in the matrix $(s_{i,j}) \in \mathbf{F}^{P \times Q}$ in such a way that $s_{i,j} \preccurlyeq s_{i+1,j}$ and $s_{i,j} \preccurlyeq s_{i,j+1}$ where they are defined. Eventually, we get $s = s_{P,Q}$ such that $s \geq p, q$. Therefore $s \geq p \geq x$ and $s \geq q \geq y$, so $x \sim y$.

Compatibility of $\sim$ with addition and multiplication is straightforward. For $p_1, q_1, p_2, q_2 \in \mathbf{F}$ such that $p_1 \sim q_1$, and $p_2 \sim q_2$, there are r_1, r_2 such that $p_1, q_1 \leq r_1$ and $p_2, q_2 \leq r_2$. Thus $p_1 + p_2 \leq r_1 + r_2$ and $q_1 + q_2 \leq r_1 + r_2$. Therefore $p_1 + p_2 \sim q_1 + q_2$, so $\sim$ is compatible with addition.

Regarding multiplication, consider any $p, q, s \in \mathbf{F}$ such that $p \sim q$. There is r such that $p, q \leq r$. By Claim 2.1 and $p \leq r$, we get $sp, sq \leq sr$ and $ps, qs \leq rs$. Therefore $sp \sim sq$ and $ps \sim qs$.

This is sufficient for compatibility with multiplication: If $p_1 \sim q_1$ and $p_2 \sim q_2$, then $p_1 p_2 \sim q_1 p_2 \sim q_1 q_2$, so $p_1 p_2 \sim q_1 q_2$ by transitivity. □

2.2 Decreasing n

Theorem 2.4. *Let $\mathbf{A}$ be an idempotent algebra with $n \times (n+1) \times m$-terms for some $n > 1, m > 0$. Then there exists m' such that $\mathbf{A}$ has $(n-1) \times n \times m'$-terms.*

Proof. Without loss of generality, we can assume that the $n \times (n+1) \times m$-terms $f_1, \ldots, f_{nm}, g_1, \ldots, g_{(n+1)m}$ are the only basic operations of $\mathbf{A}$, and $\mathbf{A}$ is free idempotent algebra generated by two symbols 0 and 1 modulo the equations describing the $n \times (n+1) \times m$-terms.

Consider the subuniverse $R \leq \mathbf{A}^{\omega}$ generated by all the infinite sequences that have the element 1 at exactly one position and the element 0 everywhere else.

Notice that R is invariant under all permutations of ω and since $\mathbf{A}$ is idempotent, every sequence in R has only finitely many nonzero values.

By $\hat{\mathbf{A}}$ we denote the free commutative monoid generated by all the non-zero elements of $\mathbf{A}$. We identify the element $0 \in \mathbf{A}$ with the neutral element in $\hat{\mathbf{A}}$. For $\bar{x} \in R$, let $\hat{x}$ denote the sum of all nonzero values of $\bar{x}$, and let $\hat{R}$ be the set $\{\hat{x} : \bar{x} \in R\}$.

Claim 2.5. *To prove the theorem, it suffices to find*

$$x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_n \in \hat{R}$$

such that $x_1 + \cdots + x_{n-1} = y_1 + \cdots + y_n$.

If that happens, we can choose large enough m' and express the elements $x_i, y_i \in \hat{\mathbf{A}}$ as follows:

$$x_i = \sum_j^n \sum_k^{m'} z_{i,j,k} \text{ for any } i = 1, \ldots, n-1,$$

$$y_j = \sum_i^{n-1} \sum_k^{m'} z_{i,j,k} \text{ for any } j = 1, \ldots, n,$$

where $z_{i,j,k} \in \mathbf{A}$ for $i = 1, \ldots n-1$, $j = 1, \ldots n, k = 1, \ldots, m'$. Since elements x_i are in $\hat{R}$, there are (nm')-ary terms f_i' such that if we put the element 1 at the position (j,k), and zeros otherwise in f_i, we get $z_{i,j,k}$. Similarly, since elements y_j are in $\hat{R}$, there are $((n-1)m')$-ary terms g_j' such that if we put 1 at the position (i,k) and zeros otherwise into the term g_j', we get $z_{i,j,k}$. So the equations of $(n-1) \times n \times m'$-terms are satisfied by terms f_i', g_j' if variables x, y are substituted by 0 and 1, respectively. Then the equations are satisfied in general, since $0, 1$ are the generators of the free algebra $\mathbf{A}$.

Every element of $\mathbf{A}$ is a binary function $t(0,1)$ on A in variables $0, 1$. We regard them as unary functions $t(1)$ where 0 is a constant and 1 is the variable. With this viewpoint, there is a multiplication on A defined as usual function composition. $(t_1 t_2)(1) = t_1(t_2(1))$. This defines a structure of monoid on A where 1 is the neutral element and 0 is an absorbing element. For $i = 1, \ldots, n$, $j = 1, \ldots, (n+1)$, $k = 1, \ldots, m$, let $b_{i,j,k} \in A$ be the element of the monoid defined by

$$b_{i,j,k} = f_i(0, 0, \ldots, 0, \underset{(j,k)}{1}, 0, \ldots, 0) = g_j(0, 0, \ldots, 0, \underset{(i,k)}{1}, 0, \ldots, 0),$$

and let $\mathbf{B}$ be the submonoid generated the elements $b_{i,j,k}$. Finally, let $\hat{\mathbf{B}} = (\hat{B}, +, \cdot, 0, 1)$ be the additive submonoid of $\hat{\mathbf{A}}$ generated by elements of $\mathbf{B}$ with multiplicative structure inherited from $\mathbf{B}$, so $\hat{\mathbf{B}}$ is the free semiring generated by elements $b_{i,j,k}$. Notice that the universe of $\hat{\mathbf{B}}$ is a subset of the universe of $\hat{\mathbf{A}}$.

We equip the semiring $\hat{\mathbf{B}}$ with equations E of the form

$$\sum_i^n \sum_k^m b_{i,j,k} = 1 \text{ for all } j = 1, \ldots, (n+1),$$

$$\sum_j^{n+1} \sum_k^m b_{i,j,k} = 1 \text{ for all } i = 1, \ldots, n.$$

In other words, these equations actually say that

$$f_i(1, 0, \ldots, 0) + f_i(0, 1, \ldots, 0) + \cdots + f_i(0, 0, \ldots, 1) = 1,$$

$$g_i(1,0,\dots,0) + g_i(0,1,\dots,0) + \dots + g_i(0,0,\dots,1) = 1.$$

Let $\sim$ be the congruence generated by these equations E.

Claim 2.6. *If $p, q \in \hat{\mathbf{B}}$ such that q is a single expansion of p using equations E and $p \in \hat{R}$, then also $q \in \hat{R}$.*

Let t be a term in ω variables (using just finitely many of them) that takes the generators of R and outputs some $\bar{r} \in R$ such that $\hat{r} = p$. We prove the claim by induction on the complexity of t. Let $p = uv + s$ and $q = uev + s$ where u, v are monomials, s is a polynomial, and e is a single expansion of 1. If $u = 1$, we prove the claim directly. Any single expansion e of 1 is of the form

$$h(1,0,\dots,0) + h(0,1,0,\dots,0) + \dots + h(0,\dots,0,1),$$

where h is a basic operation of $\mathbf{A}$. Let us denote the arity of h as k and the summands as b_i for $i = 1,\dots,k$. So we can write $e = \sum_{i=0}^{k} b_i$. We take k different representations $\bar{r}_1, \dots \bar{r}_k \in R$ that differs only in the possition of v (if there are multiple v in $\bar{r}$, we vary the position of one of them and fix the rest). Then $h(\bar{r}_1,\dots,\bar{r}_k)$ correspond to the polynomial $ev + s = q$.

If $u \neq 1$, we use the induction hypothesis. Assume that $\bar{r} = h(\bar{r}_1,\dots,\bar{r}_k)$ for an elementary operation h, where all the construction terms for $\bar{r}_1,\dots,\bar{r}_k$ are simpler. We follow the position of uv in the sequence $\bar{r}$. On that position, we see $uv = h(w_1,\dots,w_k)$. There are two possibilities. Either idempotency is applied and $w_1 = \dots = w_k$, or one more letter is appended to the word, therefore all the elements w_i except one are zeros. In the case of idempotency, we use a single expansion step to all the sequences $\bar{r}_i$ in the same way – we replace the position with uv by multiple positions covering uev. We denote these modified sequences $\bar{r}_i$ as $\bar{r}'_i$. The sequences $\bar{r}'_i$ were obtained from $\bar{r}_i$ using a single expansion step, so they are in R by induction hypothesis. Finally, $\bar{r}' = h(\bar{r}_1,\dots,\bar{r}_k) \in R$ and $q = \hat{r}'$.

In the other case, there is one non-zero $w_i = u_2 v$, where $u = u_1 u_2$ and u_1 is one of the generators of $\mathbf{B}$. Again, we replace the $u_2 v$ in R by $u_2 ev$ in $\bar{r}_i$, getting $\bar{r}'_i \in R$ by induction hypothesis. For $j \neq i$, we obtain $\bar{r}'_j$ just by expanding the number of zeros at the position of uv so that the corresponding positions still match. Finally, $\bar{r}' = h(\bar{r}'_1,\dots,\bar{r}'_k) \in R$ and $q = \hat{r}'$.

Claim 2.7. *To prove the theorem, it suffices to show that $n-1 \sim n$ in $\hat{\mathbf{B}}$.*

Indeed, if $n-1 \sim n$, there is a common expansion s by Proposition 2.1. Since s is an expansion of $n-1$, there are $x_1,\dots,x_{n-1}$ such that $\sum_i^{n-1} x_i = s$, and every x_i is an expansion of 1. Similarly, since s is an expansion of n, there are $y_1,\dots,y_{n-1}$ such that $\sum_i^{n} y_i = s$, and every y_i is an expansion of 1. Therefore all the elements $x_i, y_i \in \hat{R}$ and the assumptions of Claim 2.5 are satisfied.

Now we translated the original problem into the language of the semiring $\hat{\mathbf{B}}$ modulo $\sim$. Before general reasoning, we show the idea on the example $n = 2, m = 1$. So $\hat{\mathbf{B}}$ is generated by $b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}$, congruence $\sim$ is generated by

$$1 \sim b_{11} + b_{12} + b_{13} \sim b_{21} + b_{22} + b_{23} \sim b_{11} + b_{21} \sim b_{12} + b_{22} \sim b_{13} + b_{23},$$

and we want to prove $1 \sim 2$. Clearly $2 \sim 3$ since

$$2 \sim (b_{11}+b_{12}+b_{13})+(b_{21}+b_{22}+b_{23}) = (b_{11}+b_{21})+(b_{12}+b_{22})+(b_{13}+b_{23}) \sim 3.$$

Now, let us expand 1 a bit.

$$1 \sim b_{11}+b_{12}+b_{13} \sim b_{11}(b_{21}+b_{22}+b_{23})+(b_{11}+b_{12}+b_{13})b_{12}+(b_{11}+b_{12}+b_{13})b_{13}$$
$$= b_{11}(b_{22}+b_{12}+b_{23}+b_{13})+\cdots \sim 2b_{11}+\cdots$$

We managed to get $2b_{11}$ in the expanded 1. Since $2 \sim 3$, we get an extra b_{11}, and then collapse the expression using the reverse process. Therefore $1 \sim 1 + b_{11}$. But there is nothing special about the generator b_{11}, If we swapped $b_{11} \leftrightarrow b_{21}$, $b_{12} \leftrightarrow b_{22}$, $b_{13} \leftrightarrow b_{23}$, we would get $1 \sim 1+b_{21}$ by the same reasoning. Therefore

$$1 \sim 1 + b_{21} \sim (1 + b_{11}) + b_{21} = 1 + (b_{11} + b_{21}) \sim 2.$$

Now, let us return to the general setup with generators $b_{i,j,k}$ for $i = 1, \ldots, n$, $j = 1, \ldots (n+1)$, $k = 1, \ldots, m$, and the congruence $\sim$ is generated by

$$1 \sim \sum_{i}^{n}\sum_{k}^{m} b_{i,j,k} \text{ for all } j = 1, \ldots, (n+1),$$

$$1 \sim \sum_{j}^{n+1}\sum_{k}^{m} b_{i,j,k} \text{ for all } i = 1, \ldots, n.$$

From the equations, we derive $n \sim n+1$

$$n \sim \sum_{i}^{n}\left(\sum_{j}^{n+1}\sum_{k}^{m} b_{i,j,k}\right) = \sum_{j}^{n+1}\left(\sum_{i}^{n}\sum_{k}^{m} b_{i,j,k}\right) = n+1.$$

We fix i', j', k'. To prove that $(n-1) \sim (n-1) + b_{i',j',k'}$ it suffices to get $nb_{i',j',k'} \sim (n+1)b_{i',j',k'}$ in an expanded form of $n-1$.

In the following calculations, by $x > y$ we mean $(\exists z : x = y + z)$.

$$n-1 \sim (n-1)\sum_{j}^{n+1}\sum_{k}^{m} b_{i',j,k} > (n-1)b_{i',j',k'} + \sum_{j\neq j'}^{n+1}\sum_{k}^{m} b_{i',j,k}$$

$$= b_{i',j',k'} \cdot \sum_{i\neq i'}^{n} 1 + 1 \cdot \sum_{j\neq j'}^{n+1}\sum_{k}^{m} b_{i',j,k}$$

$$\sim b_{i',j',k'}\left(\sum_{i\neq i'}^{n}\sum_{j}^{n+1}\sum_{k}^{m} b_{i,j,k}\right) + \left(\sum_{j}^{n+1}\sum_{k}^{m} b_{i',j,k}\right)\left(\sum_{j\neq j'}^{n+1}\sum_{k}^{m} b_{i',j,k}\right)$$

$$> b_{i',j',k'}\left(\sum_{i\neq i'}^{n}\sum_{j\neq j'}^{n+1}\sum_{k}^{m} b_{i,j,k} + \sum_{j\neq j'}^{n+1}\sum_{k}^{m} b_{i',j,k}\right)$$

$$= b_{i',j',k'}\left(\sum_{j\neq j'}^{n+1}\sum_{i}^{n}\sum_{k}^{m} b_{i,j,k}\right) \sim b_{i',j',k'}\cdot\sum_{j\neq j'}^{n+1} 1 = nb_{i',j',k'}.$$

Hence $n-1 \sim n-1+b_{i,j,k}$ for any i,j,k. We finaly get the desired congruence

$$n-1 \sim n-1+b_{1,1,1} \sim n-1+b_{1,1,1}+b_{1,2,1} \sim \cdots \sim n-1+\sum_{j}^{n+1}\sum_{k}^{m} b_{1,j,k} \sim n.$$

$\square$

Corollary 2.1. *Every SD($\wedge$) variety has $(m+m)$-terms for some m.*

3 Getting to $(3+m)$-terms

In this section, we prove the following

Theorem 3.1. *Every SD($\wedge$) variety $\mathcal{V}$ has a $(3+m')$-terms for large enough m'.*

By Corollary 2.1 we know that the variety has the $(m+m)$-terms for some m, denote them f, g_1, g_2. For simplicity, we may assume that the idempotent terms f, g_1, g_2 are the only basic operations of the variety, and that they satisfy only the idempotence, $(m+m)$-equations and their consequences. Let $\mathbf{A}$ be the $\mathcal{V}$-free algebra generated by elements $0, 1$

Similarly as in the proof of Theorem 2.4, we define R_n to be a n-ary relation generated by tuples with exactly one element 1 and zeros everywhere else, where $n \in \{1, 2, \ldots, \omega\}$.

For an algebra $\mathbf{B} \in \mathcal{V}$, we define a $\mathbf{B}$-*pendant* to be any subuniverse $P \subset \mathbf{B} \times \mathbf{A}^\omega$ that is invariant under all permutations of the ω positions on $\mathbf{A}^\omega$.

For any $\mathbf{B}$-pendant P we define $P|_0, P|_1 \leq \mathbf{B}$ as follows

$$P|_0 = \{b \in \mathbf{B} : (b, (0,0,\ldots,0)) \in P\}, \quad P|_1 = \{b \in \mathbf{B} : \exists \bar{r} \in R_\omega : (b, \bar{r}) \in P\}.$$

If $P|_0$ and $P|_1$ intersect, we call the pendant P *zipped.* For a subuniverse $C \leq \mathbf{B}$ and an element $b \in \mathbf{B}$, let $C[b]$ denote the smallest $\mathbf{B}$-pendant P satisfying $C \leq P|_0$ and $\{b\} \times R_\omega \leq P$. Therefore $C = C[b]|_0$ and $b \in C[b]|_1$. Clearly, if $b \in C$, the pendant $C[b]$ is zipped since b is contained in both $C[b]|_0$ and $C[b]|_1$.

Claim 3.2. *To prove the theorem, it suffices to show that the $\mathbf{A}^3$-pendant $R_3[(0,0,0)]$ is zipped.*

Indeed, the pendant $P = R_3[(0,0,0)]$ is just R_ω viewed as a subuniverse of $\mathbf{A}^3 \times \mathbf{A}^\omega$. So when that pendant is zipped, there is a common element $\bar{r}_3 \in P|_0 = R_3$ and $\bar{r}_3 \in P|_1$. By expanding the definition of $P|_1$, we get $\bar{r}_\omega \in R_\omega$ such that $(\bar{r}_3, \bar{r}_\omega) \in P = R_\omega$. Let g_1' be the term producing $\bar{r}_3$ from the generators of R_3, g_2' be the term producing $\bar{r}_\omega$ from the generators of R_ω and f' be the term producing $(\bar{r}_3, \bar{r}_\omega)$ from the generators of R_ω. We can choose large enough m' such that g_2' uses at most first m' generators and f' uses at most first $3+m'$ of them. So we perceive g_2' as m'-ary and f' as $(3+m')$-ary. Since

$$g_1\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right)=\bar{r}_3,\quad g_2\left(\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\cdots\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}\right)=\bar{r}_\omega,$$

$$f\left(\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\cdots\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}\right)=\begin{pmatrix}\bar{r}_3\\\bar{r}_\omega\end{pmatrix},$$

the equations of $(3+m')$-terms are satisfied when we plug in $x=0$ and $y=1$. However, the elements $0,1$ are the generators of a free algebra, so the equations are satisfied in general.

Lemma 3.3. *Let $\mathbf{B}$ be an idempotent algebra, $C\leq\mathbf{B}$ its subuniverse, $b\in\mathbf{B}$ an element and P be a $\mathbf{B}$-pendant such that $C\leq P|_0$ and $b\in P|_1$. Then $(C[b])|_1\leq P|_1$.*

Proof. To see that, take an element $(b,\bar{r}_\omega)\in P$ such that $\bar{r}_\omega\in R_\omega$. Let $\bar{r}_\omega$ be of the form $(x_1,x_2,\ldots,x_n,0,0,\ldots)$ for some large enough n. Since P is invariant under permutations on $\mathbf{A}^\omega$, it contains all the elements of the form

$$(b,(0,0,\ldots,0,x_1,x_2,\ldots,x_n,0,0,\ldots))$$

We construct a homomorphism $\varphi\colon\mathbf{A}\to\mathbf{A}^n$ by mapping its generators

$$0\mapsto(0,0,\ldots,0),1\mapsto(x_1,\ldots,x_n).$$

We naturally extend φ to mapping $\mathbf{A}^\omega\to(\mathbf{A}^n)^\omega=\mathbf{A}^\omega$. Notice that φ is an endomorphism of R_ω since it maps generators of R_ω into R_ω.

To finish the proof of the lemma, we take any $b'\in C[b]|_1$ and show that $b'\in P|_1$. There is $\bar{r}_\omega\in R_\omega$ such that $(b',\bar{r}_\omega)\in C[b]$. Then $\varphi(\bar{r}_\omega)\in R_\omega$ and moreover $(b',\varphi(\bar{r}_\omega))\in P$. The latter holds since the endomorphism $\psi\colon\mathbf{B}\times\mathbf{A}^\omega$ defined by $\psi((y,x))=(y,\varphi(x))$ maps the generators of $C[b]$ into P. In particular P contains all the elements $\psi(b,(0,\ldots,0,1,0,\ldots))$ for any position of 1, and $\psi(c,(0,0,\ldots)$ for any $c\in C$. So $b'\in P$ and this finishes the proof of the lemma. ∎

Now, let h be the binary term defined as

$$h(x,y)=f(\underbrace{xx\ldots x}_{m},\underbrace{yy\ldots y}_{m}),$$

Lemma 3.4. *For any $\mathbf{B}$-pendant P and $x\in P|_0,y\in P|_1$, we have $h(x,y),h(y,x)\in P|_1$.*

Proof. Without loss of generality, we may assume that $(y, (1, 0, \dots, 0)) \in P$. If not, we use Lemma 3.3 and work with $(P|_0)[y]$ instead of P. Then the lemma follows from the identities

$$f\begin{pmatrix} x & x & \dots & x, & y & y & \dots & y \\ 0 & 0 & \dots & 0, & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0, & 0 & 1 & \dots & 0 \\ & & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0, & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0, & 0 & 0 & \dots & 0 \\ & & \vdots & & & & \vdots & \end{pmatrix} = g_2 \begin{array}{c} h(x,y) \\ \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ & & \vdots & \end{pmatrix} \end{array},$$

$$f\begin{pmatrix} y & y & \dots & y, & x & x & \dots & x \\ 1 & 0 & \dots & 0, & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0, & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \\ 0 & 0 & \dots & 1, & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0, & 0 & 0 & \dots & 0 \\ & & \vdots & & & & \vdots & \end{pmatrix} = g_1 \begin{array}{c} h(y,x) \\ \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ & & \vdots & \end{pmatrix} \end{array},$$

The columns of the identities encode such sequences in $\mathbf{B} \times \mathbf{A}^\omega$ that

- are contained in P: This is apparent from the left hand side,
- has elements $h(x, y), h(y, x)$ at their first coordinates,
- the other part is contained in R_ω: This is apparent from the right hand side.

Therefore $h(x, y), h(y, h) \in P|_1$. □

Lemma 3.5. *Let $\mathbf{B}_1, \mathbf{B}_2$ be idempotent algebras, P be a $(\mathbf{B}_1 \times \mathbf{B}_2)$-pendant. Assume that there exist $x, y \in \mathbf{B}_1$ and $u, v \in \mathbf{B}_2$ such that $(x, u), (y, u), (x, v) \in P|_0$ and $(y, v) \in P|_1$. Then P is zipped.*

Proof. The pair $(h(x, y), h(v, u))$ is in the intersection $P|_0 \cap P|_1$. Indeed, it is contained in $P|_0$ since we can write

$$\begin{pmatrix} h(x,y) \\ h(v,u) \end{pmatrix} = h \begin{pmatrix} x & y \\ v & u \end{pmatrix}.$$

Alternatively, we can use the following expansion of $(h(x, y), h(v, u))$:

$$\begin{pmatrix} h(x,y) \\ h(v,u) \end{pmatrix} = h\left(h \begin{pmatrix} x, & x \\ v, & u \end{pmatrix}, h \begin{pmatrix} y, & y \\ v, & u \end{pmatrix} \right).$$

By Lemma 3.4 used twice, the pair is also an element of $P|_1$, which completes the proof. □

Lemma 3.6. *Let $\mathbf{B}_1, \mathbf{B}_2$ be idempotent algebras and $R \leq \mathbf{B}_1 \times \mathbf{B}_2$ be a compatible relation. Assume that there are elements $x \in \mathbf{B}_1, u, v \in \mathbf{B}_2$ such that $(x,u),(x,v) \in R$. Then for any $y \in \mathbf{B}_1$ the $(\mathbf{B}_1 \times \mathbf{B}_2)$-pendant $R[(y,u)]$ is zipped if and only if the $(\mathbf{B}_1 \times \mathbf{B}_2)$-pendant $R[(y,v)]$ is zipped.*

Proof. It suffices to show the forward implication. Since $R[(y,u)]$ is zipped, there is some $(y_0,u_0) \in R \cap R[(y,u)]|_1$. Consider the 4-ary relation

$$R' = \{(a_1,a_1,a_2,a_2) : (a_1,a_2) \in R\},$$

and the $(\mathbf{B}_1^2 \times \mathbf{B}_2^2)$-pendant $P = R'[(x,y,u,v)]$. Since $(y_0,u_0) \in R[(y,u)]|_1$, we can find a quadruple (x_0,y_0,u_0,v_0) in $P|_1$ for some additionally generated elements x_0, v_0. So

$$(x_0,u_0) \in R[(x,u)]|_1, \quad (x_0,v_0) \in R[(x,v)]|_1, \quad (y_0,v_0) \in R[(y,v)]|_1.$$

Since $(x,u),(x,v) \in R$, also $(x_0,u_0),(x_0,v_0) \in R$. Let Q be the pendant $R[(y,v)]$. We have $(x_0,u_0).(x_0,v_0).(y_0,u_0) \in Q|_0$ and $(y_0,v_0) \in Q|_1$. Therefore, the pendant Q is zipped by Lemma 3.5. $\square$

Finally, we define a relation $\ltimes$ on $\mathbf{A}$ as follows. We write $x \ltimes y$ if there are $u, v \in R_3$ such that

- $(x,u,v) \in R_3$,
- The $\mathbf{A}^3$-pendant $R_3[(y,u,v)]$ is zipped.

Notice that $\ltimes$ is reflexive: Indeed for any x, there are u, v such that $(x,u,v) \in R_3$. Then also $R[(x,u,v)]$ is zipped, so $x \ltimes x$.

Lemma 3.7. *If $x \ltimes y$ and there are $c, x', y' \in \mathbf{A}$ such that the triples $(x,c,x'),(y,c,y')$ are in R_3, then $x' \ltimes y'$.*

Proof. Consider u, v as in the definition of the relation $\ltimes$. We will show that $x' \ltimes y'$ by finding approptiate u', v'. We set $u' = c, v' = y$, so the condition (ii) is satisfied since $(y',c,y) \in R_3$ by symmetry of R_3. To establish $x \ltimes y$ we need to prove that $R_3[(x',c,y)]$ is zipped, equivalently, that $R_3[(y,c,x')]$ is zipped. We interpret $\mathbf{A}^3$ as $\mathbf{A} \times \mathbf{A}^2$ and use Lemma 3.6. We plug in

$$x \mapsto x, y \mapsto y, u \mapsto (u,v), v \mapsto (c,x'),$$

Indeed $(x,u,v),(x,c,x') \in R_3$ and $R_3[(y,u,v)]|$ is zipped. So the assumptions of Lemma 3.6 are satisfied, and consequently $R[(y,c,x')]$ is zipped. $\square$

We are finally ready to prove the theorem. We start with $g_1(100\ldots0) \ltimes f(100\ldots0)$ and get to $1 \ltimes h(1,0)$ using Lemma 3.7 and the following triples in R_3:

$$\begin{pmatrix} g_1(10\ldots) \\ g_1(010\ldots) \\ g_1(0011\ldots) \end{pmatrix} \begin{pmatrix} f(10\ldots) \\ f(010\ldots) \\ f(0011\ldots) \end{pmatrix}, \begin{pmatrix} g_1(0011\ldots) \\ 0 \\ g_1(1100\ldots) \end{pmatrix} \begin{pmatrix} f(0011\ldots) \\ 0 \\ f(1100\ldots) \end{pmatrix},$$

$$\begin{pmatrix} g_1(110\dots) \\ g_1(0010\dots) \\ g_1(0001\dots) \end{pmatrix} \begin{pmatrix} f(110\dots) \\ f(0010\dots) \\ f(0001\dots) \end{pmatrix}, \dots, \begin{pmatrix} g_1(0\dots0) \\ 0 \\ g_1(1\dots1) \end{pmatrix} \begin{pmatrix} h(0,1) \\ 0 \\ h(1,0) \end{pmatrix}.$$

So, there are u, v such that $(1, u, v) \in R_3$ and $R_3[(h(1,0), u, v)]$ is zipped. The first condition enforces $u = v = 0$, so $R_3[(h(1,0), 0, 0)]$ is zipped. However, by Lemma 3.4 $(h(1,0), 0, 0) \in R_3[(0,0,0)]|_1$, so $R_3[(0,0,0)]$ is zipped (Lemma 3.3, universality of pendant construction), and the proof is finished by Claim 3.2.

4 A counterexample for $(2+m)$-terms

Based on the result of the previous chapter that some $(3+m)$-terms are satisfied in every $SD(\wedge)$ variety, one could ask whether the result could be strengthened to $(2+m)$-terms. However, as we demonstrate in this section, such a generalization is not possible. Not only that there is an algebra in a $SD(\wedge)$ variety that does not have $(2+m)$-terms but there is even such an algebra that belongs to a congruence distributive variety.

Even stronger Maltsev condition than congruence distributivity is the existence of a near unanimity term. A *near unanimity* term (NU term for short) is a term t satisfying

$$t(x, x, \dots, x, \underset{i}{y}, x, \dots, x) = x$$

for any position i.

There is no algebra having an NU term and no $(2+m)$-terms, since putting g_2 to be the NU term and f, g_1 to be just the projections on the first coordinate meet the requirements of the $(2+m)$-terms. However, in our first example we demonstrate that one existence of an NU term does not imply $(2+m)$-terms for a fixed m.

Consider the following symmetric n-ary operations $t_n^{\mathbf{A}}, t_n^{\mathbf{B}}$ for $n \geq 5$ on rational numbers: Let $x_1 \leq x_2 \cdots \leq x_n$ be a sorted input of such an operation. Then

$$t_n^{\mathbf{A}}(x_1, \dots, x_n) = \frac{x_2 + \cdots + x_{n-1}}{n-2}, \quad t_n^{\mathbf{B}}(x_1, \dots, x_n) = \frac{x_3 + \cdots + x_{n-2}}{n-4}.$$

If the input is not sorted, we first sort it and then perform the calculation. These operations are clearly NU, that is,

$$t_n^{\mathbf{A}}(x, x, \dots, x, y, x, \dots x) = t_n^{\mathbf{B}}(x, x, \dots, x, y, x, \dots x) = x$$

for any position of y.

For proving key properties of t, we need a lemma.

Lemma 4.1. *Let $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{Q}$ be such that $x_i \leq y_i$ for all $i = 1, \dots, n$. Let $x'_1, \dots, x'_n$ be $x_1, x_2, \dots, x_n$ sorted in increasing order, and let $y'_1, \dots, y'_n$ be sorted $y_1, \dots y_n$. Then $x'_i \leq y'_i$ for all i and the set $\{i : x'_i < y'_i\}$ is at least as large as the set $\{i : x_i < y_i\}$.*

Proof. Without loss of generality, let the numbers x_i be increasing in lexicographical order. Therefore $x_i = x'_i$ for all i. It is possible to sort the sequence y_i by consecutive application of *sorting transpositions*, that is swaping y_i with y_j if $i < j$ and $y_i > y_j$. An example of such an process is the well known bubble sort algorithm. We show that one sorting transposition preserves the condition $x_i \le y_i$ for all i, and does not shrink the set $\{i : x_i < y_i\}$. In one such transposition, the swapped positions i, j are independent of all the others, so we may assume that there are no others. In particular $n = 2$, $x_1 \le x_2$, $y_1 > y_2$, $x_1 \le y_1$, $x_2 \le y_2$, $y'_1 = y_2$, $y'_2 = y_1$. First $x_1 \le x_2 \le y_2$ and $x_2 \le y_2 < y_1$, so $x_1 \le y'_1$ and $x_2 < y'_2$. This shows that $x_i \le y_i$ for all i. Now, let us investigate the number of strict inequalities. Since $x_2 < y'_2$, the size of the set $\{i : x'_i < y'_i\}$ is at least 1. If the size equals two, we are done. Otherwise $x_1 = y'_1$, so $x_1 = x_2 = y_2$. Since $x_2 = y_2$, the size of the set $\{i : x_i < y_i\}$ is at most one, so it is not larger than $\{i : x'_i < y'_i\}$. ☐

Claim 4.2. *For any $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{Q}$ such that $x_i \le y_i$ for all i, we have $t^{\mathbf{A}}_n(x_1, \dots, x_n) \le t^{\mathbf{A}}_n(y_1, \dots, y_n)$. The inequality is strict if $x_i < y_i$ for at least three i.*

Indeed, we can assume that x_i and y_i are sorted by Lemma 4.1. The first part is then clear from definition of $t^{\mathbf{A}}$. If $x_i < y_i$ for at least three i, it happens for at least one $i \ne 1, n$, and that $x_i < y_i$ causes the strict inequality.

Consider the algebras $\mathbf{A}_n = (\mathbb{Q}, t^{\mathbf{A}}_n)$, $\mathbf{B}_n = (\mathbb{Q}, t^{\mathbf{B}}_n)$. For $m \ge 1$ define the sets $U \subset \mathbb{Q}^2, V_m \subset \mathbb{Q}^m, W_m \subset \mathbb{Q}^{2+m}$ as follows:

$$U = \{(a_1, a_2) : a_1 + a_2 = 1\},$$

$$V_m = \{(b_1, \dots, b_m) : b_1 \dots b_m \ge 0 \text{ and there is a nonzero } b_i.\}$$

$$W_m = \{(a_1, a_2, b_1, \dots, b_m) :$$

$$(a_1 + a_2 < 1 \text{ and } b_1 \dots b_m \ge 0) \text{ or } (a_1 + a_2 = 1 \text{ and } b_1 = \dots = b_m = 0))\}$$

Claim 4.3. *For any $n \ge 5$, the set U is a subuniverse of $\mathbf{A}^2_n$.*

The claim follows from the fact that if $x_1, x_2, \dots, x_n$ is non-decreasing, then also $1 - x_n, \dots, 1 - x_2, 1 - x_1$ is non-decreasing.

Claim 4.4. *For any $n \ge 5$, $2m < n$ the set V_m is a subuniverse of $\mathbf{B}^m_n$.*

Indeed, if at least three of $x_1, \dots, x_n$ are non-zero, then $t^{\mathbf{B}}$ is also non-zero. Consider m-tuples $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$. Every m-tuple $\bar{x}_i$ has a non-zero position p_i. Since $2m < n$, one of the positions has to repeat three times, $p = p_{i_1} = p_{i_2} = p_{i_3}$. So the m-tuple $t(\bar{x}_1, \dots \bar{x}_n)$ has a non-zero element at the position p.

Claim 4.5. *For any $m \ge 1, n \ge 5$, the set W_m is a subuniverse of $\mathbf{A}^2_n \times \mathbf{B}^m_n$.*

For the same reason as in Claim 4.3, the projection of W_i to $\mathbf{A}^2$ is a subuniverse of $\mathbf{A}^2$. The question is about subtle detail how it interacts with the $\mathbf{B}^m$-part. Let us take $(2 + m)$-tuples $\bar{x}_1, \dots, \bar{x}_n \in W_m$ and show that $t(\bar{x}_1, \dots, \bar{x}_n)$ belongs to W_m as well. Let $a_{i,j}, b_{i,j}$ be matrices such that $\bar{x}_j = (a_{1,j}, a_{2,j}, b_{1,j}, \dots, b_{m,j})$. We analyze two cases:

1. For at most two columns j it happens that $a_{1,j} + a_{2,j} < 1$. Then all the other columns have zero $\mathbf{B}^m$-part, so $t^{\mathbf{B}}(b_{i,1}) = 0$ for any $\mathbf{B}$-row i. Hence $t(\bar{x}_1, \dots, \bar{x}_n) \in W_m$.

2. For at least three columns j it happens that $a_{1,j} + a_{2,j} < 1$. In other words, at these three positions j it happens that $a_{1,j} < 1 - a_{2,j}$ while non-strict inequality is satisfied everywhere. Thus, by Claim 4.2, we have

$$t^{\mathbf{A}}(a_{1,1}, \dots, a_{1,n}) < t^{\mathbf{A}}(1 - a_{2,1}, \dots, 1 - a_{2,n}) = 1 - t^{\mathbf{A}}(a_{2,1}, \dots, a_{2,n}).$$

   Equivalently,

$$t^{\mathbf{A}}(a_{1,1}, \dots, a_{1,n}) + t^{\mathbf{A}}(a_{2,1}, \dots, a_{2,n}) < 1,$$

   so $t(\bar{x}_1, \dots, \bar{x}_n)$ belongs to W_i.

So, in both cases, the result belongs to W_m, and the claim is established.

We are now ready to construct the counterexamples.

Theorem 4.6. *For any n, m such that $n \geq 5$ and $2m < n$, there is an algebra having an n-ary NU-term, $n \geq 5$, but no $(2+m)$-terms.*

Proof. The algebra is $\mathbf{C}_n = \mathbf{A}_n \times \mathbf{B}_n$. For a contradiction, suppose that $\mathbf{C}_n$ has $(2+m)$-terms f, g_1, g_2. These terms are common for all the algebras in the variety generated by $\mathbf{C}$. In particular, there are operations $g_1^{\mathbf{A}}, g_2^{\mathbf{A}}, f^{\mathbf{A}}$ on $\mathbf{A}_n$ and $g_1^{\mathbf{B}}, g_2^{\mathbf{B}}, f^{\mathbf{B}}$ on $\mathbf{B}_n$ such that

$$\begin{aligned}
g_1^{\mathbf{A}}(1,0) &= f^{\mathbf{A}}(1,0,0,0,0\dots,0,0) = a_1,\\
g_1^{\mathbf{A}}(0,1) &= f^{\mathbf{A}}(0,1,0,0,0\dots,0,0) = a_2,\\
g_2^{\mathbf{B}}(1,0,0,\dots,0,0) &= f^{\mathbf{B}}(0,0,1,0,0\dots,0,0) = b_1,\\
g_2^{\mathbf{B}}(0,1,0,\dots,0,0) &= f^{\mathbf{B}}(0,0,0,1,0\dots,0,0) = b_2,\\
&\vdots\\
g_2^{\mathbf{B}}(0,0,0,\dots,0,1) &= f^{\mathbf{B}}(0,0,0,0,0\dots,0,1) = b_m.
\end{aligned}$$

The tuple $(a_1, a_2, b_1, \dots, b_n)$ belongs to W_m since W_m contains all the columns on the right hand side. Similarly, $(a_1, a_2) \in U$ and $(b_1, \dots, b_m) \in V_m$ by left hand side. But there are is no such tuple W_m that is composed from the tuples in U and V_m. $\square$

Theorem 4.7. *There is an algebra in a congruence distributive variety that has no $(2+m)$-terms.*

Proof. The proof is similar, we take the algebra $\mathbf{C}_6 = \mathbf{A}_6 \times \mathbf{B}_6$. We just modify it a bit in order to make V_m a subuniverse for any m. Let s be the following 4-ary minor of t

$$s(x, y, z, w) = t(x, y, z, w, w, w).$$

Consider the algebra $\mathbf{C}' = (\mathbb{Q}^2, s^{\mathbf{C}})$. The algebra $\mathbf{C}'$ is congruence distributive, since it has the following directed Jónsson terms written as minors of the term s:

$$\begin{aligned}
s(xyzz) &= t(xyzzzz),\\
s(xxyz) &= t(xxyzzz),\\
s(zzyx) &= t(xxxyzz),\\
s(zyxx) &= t(xxxxyz).
\end{aligned}$$

For the definition of directed Jónsson terms, we refer the reader to [6].

On the other hand, $\mathbf{C}'$ does not have any $(2+m)$-terms. For a contradiction, let us assume that there are term operations $f^{\mathbf{C}}, g_1^{\mathbf{C}}, g_2^{\mathbf{C}}$ in the algebra $\mathbf{C}$. So there are such terms even in $\mathbf{A}' = (\mathbb{Q}, s^{\mathbf{A}})$ and $\mathbf{B}' = (\mathbb{Q}, s^{\mathbf{B}})$. We consider the same $2 + m$ equalities as in the previous proof, resulting in $a_1, a_2, b_1, \ldots, b_m$. Since the basic operations of algebras $\mathbf{A}', \mathbf{B}'$ are defined from the operations of the algebras $\mathbf{A}, \mathbf{B}$, the set U is still a subuniverse of $(\mathbf{A}')^2$ and the set W_m is still a subuniverse of $(\mathbf{A}')^2 \times (\mathbf{B}')^m$. So $(a_1, a_2) \in U$ and $(a_1, a_2, b_1, \ldots, b_m) \in W_m$. We cannot directly use Claim 4.4 to ensure that V_m is a subuniverse of $(\mathbf{B}')^m$ since the claim assumes $2m < 6$. However, it is still true. We can check it manually: If $\bar{x}, \bar{y}, \bar{z}, \bar{w} \in V$ and $w_i > 0$ for some i, then even

$$s^{\mathbf{B}}(x_i, y_i, z_i, w_i) = t^{\mathbf{B}}(x_i, y_i, z_i, w_i, w_i, w_i) > 0,$$

so $s^{\mathbf{B}}(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ has a non-zero position. Therefore V_m is a subuniverse of $(\mathbf{B}')^m$, $(b_1, \ldots, b_m) \in V_m$, and we get the same contradiction as in the previous proof. $\square$

5 Further work

Since Question 1.1 remained open, the main objective is still to find out whether or not the $SD(\wedge)$ property is characterized by a strong Maltsev condition. The $(3 + n)$-terms are general enough for $SD(\wedge)$ while the $(2 + n)$-terms are too strong. Therefore we suggest $(3+3)$-terms as the candidate for a strong Maltsev condition, or a good starting point for proving the opposite.

Question 5.1. *Is there a $SD(\wedge)$ variety that does not have $(3 + 3)$-terms?*

It is also reasonable to start with a stronger property than congruence meet-semidistributivity, namely simple congruence distributivity, or the one in Theorem 1.3.

Question 5.2. *Are $(3 + 3)$-terms implied by*

(a) directed Jónsson terms? (equivalent to congruence distributivity, see [6])

(b) ternary and 4-ary weak NU terms $w_3.w_4$ *such that* $w_3(y, x, x) = w_4(y, x, x, x)$*?*

Miklós Maróti with Ralph McKenzie (see Theorem 1.3 of [10]) proved that congruence distributivity implies the existence of all at least ternary weak NU terms. However, the catalogue of counterexamples is so weak, that even the "glued" weak NU terms, as in item (b), are still plausible candidates for the strong Maltsev condition too. On the other hand, congruence distributivity is the weakest general condition under which we know about the weak NU terms. So we ask the following.

Question 5.3. *Is the existence of a weak NU term implied by the* $SD(\wedge)$ *property? In particular, is it implied by* $(3+3)$*-terms?*

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