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Loop conditions with strongly connected graphs

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Abstract

We prove that the existence of a term s satisfying $s(r, a, r, e) = s(a, r, e, a)$ in a general algebraic structure is equivalent to an existence of a term t satisfying $t(x, x, y, y, z, z) = t(y, z, z, x, x, y)$. As a consequence of a general version of this theorem and previous results we get that each strongly connected digraph of algebraic length one, which is compatible with an operation t satisfying an identity of the from $t(\dots) = t(\dots)$, has a loop.

1 Introduction

Under which structural and algebraic conditions does a graph compatible with an algebra necessarily have a loop? This question has originated from the algebraic approach to the fixed–template constraint satisfaction problems and answers have provided strong results and useful tools in this area as well as in universal algebra, see [4, 2] for recent surveys.

A. Bulatov's algebraic refinement of the well–known Hell and Nesetril dichotomy for undirected graphs [9] can be stated as follows.

Theorem 1.1 (basic loop lemma)**.** *[8] If a finite undirected graph* $\mathbb{G}$

- *contains a triangle and*
- *is compatible with a Taylor term (that is, an operation satisfying a nontrivial set of identities of the form $t(\textit{some vars}) = t(\textit{some vars})$),*

then $\mathbb{G}$ contains a loop (that is an edge joining a vertex with itself).

In 2010, M. Siggers observed that this structural result has an algebraic equivalent statement.

Theorem 1.2. *[13] If* $\mathbf{A}$ *is a finite algebra with a Taylor term, then* $\mathbf{A}$ *has a 6-ary Siggers term s satisfying $s(x, x, y, y, z, z) = s(y, z, z, x, x, y)$.*

Proof. Let $\mathbf{F}$ be the $\mathbf{A}$-free algebra generated by the set $\{x, y, z\}$. Since $\mathbf{A}$ is finite, $\mathbf{F}$ is finite as well. We consider a symmetric binary relation (graph) on $\mathbf{F}$ generated by the pairs

$$\binom{x}{y}, \binom{x}{z}, \binom{y}{z}, \binom{y}{x}, \binom{z}{x}, \binom{z}{y}.$$

This graph has a triangle on x, y, z, so it has a loop (l, l) by the loop lemma. Let s be the six-ary term that generates the loop from the generators above. Then the following identity holds for the generators x, y, z.

$$s(x, x, y, y, z, z) = l = s(y, z, z, x, x, y)$$

However, since $\mathbf{F}$ is a free algebra, the identity holds in general. □

Notice that obtaining the structural result from the algebraic one is even easier – if we have a Siggers term s, and an undirected triangle in a compatible graph, we find the loop directly by applying the Siggers term to the six edges of the triangle (we use both directions of each of three undirected edge).

The loop lemma was improved by L. Barto and M. Kozik.

Theorem 1.3 (loop lemma)**.** *[3] If a finite digraph* $\mathbb{G}$

- *is weakly connected,*
- *is smooth (has no sources and no sinks),*
- *has algebraic length 1 (cannot be mapped to a non-trivial directed cycle) and*
- *is compatible with a Taylor term,*

then $\mathbb{G}$ *contains a loop.*

Consequently, one can improve Theorem 1.3 as follows.

Theorem 1.4. *[11] If* $\mathbf{A}$ *is a finite algebra with a Taylor term, then* $\mathbf{A}$ *has a 4-ary Siggers term* s *satisfying* $s(r, a, r, e) = s(a, r, e, a)$.

Since both algebraic results give a Taylor term, the following properties are equivalent for a finite algebra $\mathbf{A}$:

(i) $\mathbf{A}$ has a Taylor term.

(ii) $\mathbf{A}$ has a 6-ary Siggers term $s(x, x, y, y, z, z) = s(y, z, z, x, x, y)$.

(iii) $\mathbf{A}$ has a 4-ary Siggers term $s(r, a, r, e) = s(a, r, e, a)$.

A natural question is whether the equivalence can be generalized to general (infinite) algebras. A. Kazda found an example [10] of an algebra that has an idempotent Taylor term (that is, a Taylor term t which additionally satisfies t(x,....x)=x), but no 6-ary or 4-ary Siggers term. Therefore the remaining question is whether the properties (ii) and (iii) are equivalent.

Recently, the author studied such conditions in general [12]. A *loop condition* is a condition of the form: There is a term t satisfying t(some variables) = t(some variables). A consequence of results in [12] is that the following statements are equivalent for every algebra $\mathbf{A}$.

(i) $\mathbf{A}$ satisfies a non-trivial loop condition,

(ii) $\mathbf{A}$ has a 6-ary Siggers term $s(x,x,y,y,z,z) = s(y,z,z,x,x,y)$.

(iii) Every non-bipartite undirected graph compatible with an algebra $\mathbf{B}$ in the variety generated by $\mathbf{A}$ has a loop.

In this paper, we give a positive answer to the raised question by finding a broader class of loop conditions that are equivalent. This allows us to give an even stronger structural characterization of algebras satisfying some loop condition in the following form.

Theorem 1.5. *Let $\mathcal{V}$ be a variety. Then the following are equivalent.*

(i) $\mathcal{V}$ satisfies a non-trivial loop condition,

(ii) Let $\mathbb{G}$ be a graph compatible with $\mathbf{A} \in \mathcal{V}$. If $\mathbb{G}$ has a strongly connected component with algebraic length one, then $\mathbb{G}$ has a loop.

1.1 Outline

We start by providing a proper definition of the key concepts in Section 2 including a digraph associated to a loop condition. We also summarize there the results of the paper [12].

In Section 3 we prove our main result: that all loop conditions with a strongly connected digraph with algebraic length one are equivalent.

Then we give two examples of classical classes that satisfy such a condition in Section 4.

In Section 5, we classify the strength of loop comditions with strongly connected digraphs (of arbitrary algebraic lengths)

Finally, in Section 6 we discuss the case whithout strong connectedness. This case seems to be much colorful, so we provide at least some counterexamples and some partial results.

2 Preliminaries

We refer to [6, 7] for undefined notions and more background.

2.1 Digraphs, algebraic length

A *digraph* $\mathbb{G} = (A, G)$ is a relational structure, where A is the set of nodes (vertices) and $G \subset A \times A$ is a binary relation, in other words, the set of edges. We sometimes denote an edge $(x, y) \in G$ of a digraph by

$$x \overset{\mathbb{G}}{\to} y, \text{ or } y \overset{\mathbb{G}}{\leftarrow} x.$$

If both $(x, y), (y, x) \in G$, we write $x \overset{\mathbb{G}}{-} y$. If the digraph is clear from the context, we may write just $x \to y, y \leftarrow x$, or $x - y$. A loop in $\mathbb{G}$ is an edge $a \to a$.

If the relation G is symmetric, we also call $\mathbb{G}$ as undirected graph.

Consider two digraphs $\mathbb{G} = (A, G)$ and $\mathbb{H} = (B, H)$. A digraph homomorphism $f \colon \mathbb{G} \to \mathbb{H}$ is a mapping $A \to B$ such that $f(a_1) \overset{\mathbb{H}}{\to} f(a_2)$ whenever $a_1 \overset{\mathbb{G}}{\to} a_2$.

Thorough the paper, we use the following basic digraphs:

1. A clique $\mathbb{K}_n = (\{0, \dots, n-1\}, \neq)$,
2. a symmetric cycle $\mathbb{C}_n$ on $\{0, \dots, n-1\}$, where $x - y$ if $y \equiv x \pm 1 \pmod n$,
3. a directed cycle $\mathbb{D}_n$ on $\{0, \dots, n-1\}$, where $x \to y$ if $y \equiv x + 1 \pmod n$,
4. a directed path $\mathbb{Z}$ on integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$, where $k \to k+1$ for all integers k.

An *oriented walk* of length n is a sequence of vertices $x_0, x_1, \dots x_n$ such that for each i we have $x_i \to x_{i+1}$ or $x_i \leftarrow x_{i+1}$. A walk is called *directed* if for all i we have the forward edge $x_i \to x_{i+1}$. If $x_0 = x_n$, we also talk about directed or oriented cycle. Notice that a directed cycle as a walk of length n on a digraph $\mathbb{G}$ corresponds to a digraph homomorphism $\mathbb{D}_n \to \mathbb{G}$.

If every pair of vertices is connected by an oriented walk, we call the digraph *weakly connected*. If every pair of vertices is connected by a directed walk (in both ways), we call the digraph *strongly connected*.

Algebraic length of an oriented walk is defined as the number of forward edges minus the number of backward edges in the walk. The *algebraic length of a digraph* $\mathbb{G}$, denoted $al(\mathbb{G})$, is the greatest common divisor of the algebraic lengths of all oriented cycles, or ∞ if all oriented cycles have algebraic length zero. Note that our definition slightly differs from the one from [3] on digraphs that are not weakly connected. On the other hand, neither of papers is interested in that case. If $\mathbb{G}$ is weakly connected, then there is an oriented $\mathbb{G}$-cycle of length $al(\mathbb{G})$.

There are also nice characterizations of these properties using digraph homomorphism. The algebraic length of a digraph $\mathbb{G}$ is the biggest n such that there is a digraph homomorphism $\mathbb{G} \to \mathbb{D}_n$, or ∞ if there is a digraph homomorphism $\mathbb{G} \to \mathbb{Z}$. Conversely, a finite digraph $\mathbb{G}$ is strongly connected if and only if there is a digraph homomorphism $\mathbb{D}_n \to \mathbb{G}$ for some n that is surjective on edges. We finish this subsection with an even stronger property of finite strongly connected digraphs with a given algebraic length.

Proposition 2.1. *Let $\mathbb{G}$ be a finite strongly connected digraph and let S be the set of all numbers n such that there is a digraph homomorphism $\mathbb{D}_n \to \mathbb{G}$ that is surjective on edges. Then S contains only the multiples of $al(\mathbb{G})$ but it contains any such a multiple that is large enough.*

Proof. Let d denote the greatest common divisor of elements of S. The set S is closed under addition: Any two directed cycles that covers all edges can be joined at any node. Therefore, by Schur's theorem, the set S contains all the multiples of d that are large enough. Every number of S expresses an algebraic length of a directed cycle, so $al(\mathbb{G}) \mid d$. It remains to prove that $d \mid al(\mathbb{G})$.

To see that, take $k \in S$, the corresponding directed cycle c_k and an oriented cycle c_{al} on $\mathbb{G}$ of algebraic length $al(\mathbb{G})$. In c_{al}, we append one copy of c_k and replace every backward edge by $(k-1)$ forward edges taken from c_k. The result is a directed cycle c that is surjective on edges and its length is equal to $al(\mathbb{G}) + nk$ for some n. Hence

$$al(\mathbb{G}) + nk \in S.$$

Since $d \mid k$ and $d \mid al(\mathbb{G}) + nk$, also $d \mid al(\mathbb{G})$ and we are done. □

2.2 Compatible digraphs, pp-constructions

An n-ary operation $f : A^n \to A$ is said to be *compatible* with an m-ary relation $R \subset A^m$ if $f(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) \in R$ for any $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n \in R$. Here (and later as well) we abuse notation and use f also for the n-ary operation on A^m defined coordinate-wise.

An algebra $\mathbf{A} = (A, f_1, f_2, \ldots)$ is said to be compatible with a relational structure $\mathbb{A} = (A, R_1, R_2, \ldots)$ if all the operations $f_1, f_2, \ldots$ are compatible with all the relations $R_1, R_2, \ldots$.

We will extensively use a standard method for building compatible relations from existing ones – primitive positive (*pp*, for short) definitions. A relation R is *pp-definable* from relations $R_1, \ldots, R_n$ if it can be defined by a first order formula using variables, existential quantifiers, conjunctions, the equality relations, and predicates $R_1, \ldots, R_n$. Clauses in pp-definitions are also referred to as *constraints*. Recall that if $R_1, \ldots, R_n$ are compatible with an algebra, then so is R.

A pp-definition of a k-ary relation R from a digraph $\mathbb{G}$ can be described by a finite digraph $\mathbb{H}$ with k distinguished vertices $v_1, \ldots, v_k$. We define $R(x_1, \ldots, x_k)$ by the existence of a digraph homomorphism $\mathbb{H} \to \mathbb{G}$ that maps v_i to x_i. The edges of $\mathbb{H}$ correspond to the constraints in the pp-definition, and the images of remaining vertices of $\mathbb{H}$ (other that $v_1, \ldots, v_k$) are existentialy quantified.

Using a more general technique, we can also construct a relation that is compatible with a subalgebra of a power $\mathbf{A}^k$. We pp-define two relations from some relations compatible with $\mathbf{A}$: a k-ary relation B, and a nk-ary relation

R. Then B is compatible with $\mathbf{A}$, so it forms a subuniverse of $\mathbf{A}^k$. The nk-ary relation R is perceived as an n-ary relation on elements of $\mathbf{A}^k$, so R acts as a n-ary relation on B as well. Even in this perception, the relation R is still compatible with the subalgebra $\mathbf{B} \leq \mathbf{A}^k$ on subuniverse B. This construction is a special case of pp-construction.

In our proof, we will use this construction in the following form. Consider two finite (template) digraphs $\mathbb{V} = (V_v, V_e)$, $\mathbb{E} = (E_v, E_e)$, and two digraph homomorphisms $\phi_0, \phi_1 \colon \mathbb{V} \to \mathbb{E}$. Then we take a digraph $\mathbb{G} = (A, G)$ compatible with an algebra $\mathbf{A}$, and construct a digraph $\mathbb{H} = (B, H)$ where B is the set of all the mappings $\mathbb{V} \to \mathbb{G}$ and edges defined as follows: The is an edge $v_0 \overset{\mathbb{H}}{\to} v_1$ if there is a homomorphism $e \colon \mathbb{E} \to \mathbb{G}$ such that $v_0 = e \circ \phi_0$ and $v_1 = e \circ \phi_1$. Then B is a subuniverse of $\mathbf{A}^{|V_v|}$, and H a compatible digraph with the appropriate algebra $\mathbf{B}$.

2.3 Loop conditions

A *variety* $\mathcal{V}$ is a class of algebras closed under powers, subalgebras and homomorphic images. In every variety, for any set S, we can find the *free algebra* $\mathbf{F}$ *freely generated by* S with the following property: Let $t_1.t_2$ be any n-ary term operations, and $s_1, \ldots, s_n$ be distinct elements of S. If $t_1(s_1 \ldots, s_n) = t_2(s_1, \ldots, s_n)$ in $\mathbf{F}$, then $t_1(x_1 \ldots, x_n) = t_2(x_1, \ldots, x_n)$ for any $x_1, \ldots, x_n$ in any algebra from $\mathcal{V}$.

A *loop condition* is given by a set of variables V and two n-tuples

$$x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in V.$$

An algebra (or variety) is said to satisfy such a condition if there is a term t in the algebra (variety) satisfying the identity

$$t(x_1, x_2, \ldots, x_n) = t(y_1, y_2, \ldots, y_n).$$

To such a loop condition C, we assign a digraph

$$\mathbb{G}_C = (V, \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$$

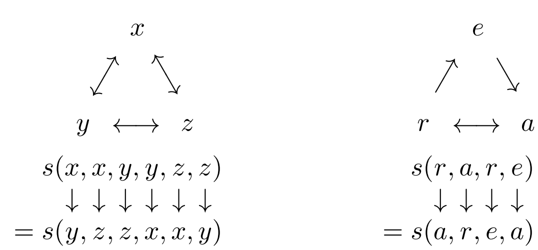

Figure 1: Graphs corresponding to the 6-ary and 4-ary Siggers terms

Since $\mathbf{A}$ satisfies a certain loop condition if and only if the variety generated by $\mathbf{A}$ satisfies it, it suffices to focus on varieties.

Proposition 2.2. *The following are equivalent for a variety* $\mathbb{V}$ *and a loop condition* C*.*

1. $\mathcal{V}$ *satisfies* C*.*

2. *Let* $\mathbf{F}$ *be the free algebra freely generated by the set of variables in* C*, and let* $\mathbb{G}$ *be a digraph compatible with* $\mathbf{F}$ *containing all the edges of* $\mathbb{G}_c$*. Then* $\mathbb{G}$ *have a loop.*

3. *For any algebra* $\mathbf{A} \in \mathcal{V}$ *and any digraph* $\mathbb{G}$ *compatible with* $\mathbf{A}$*, if there is a digraph homomorphism* $\mathbb{G}_C \to \mathbb{G}$*, then* $\mathbb{G}$ *has a loop.*

The proof of the proposition is just a direct generalization of the proof of Theorem 1.2. For a general proof, we refer the reader to [12].

It is apparent from the proposition that a validity of a loop condition C in a variety (algebra) is determined by the digraph $\mathbb{G}_C$. Based on that tight connection between a loop condition and its digraph, we sometimes assign digraph attributes (strongly connected, algebraic length) to a loop condition, meaning the attributes of the corresponding digraph. For a general finite digraph $\mathbb{G}$ we also talk about the $\mathbb{G}$ loop condition – that refers to any loop condition C such that $\mathbb{G}_C \simeq \mathbb{G}$.

A loop condition is called *trivial* if it is satisfied in every algebra, equivalently if its digraph contains a loop. Otherwise, the loop condition is called *non-trivial*.

A simple reason for implication between loop conditions, directly obtained from item (iii) of Proposition 2.2, is the following theorem.

Proposition 2.3. *If there is a digraph homomorphism* $\mathbb{G} \to \mathbb{H}$*, then every variety (algebra) satisfying the* $\mathbb{G}$ *loop condition satisfies the* $\mathbb{H}$ *loop condition as well.*

Nevertheless, this is not by far the only way for obtaining an implication between loop conditions. It is proved in [12] that all non-trivial undirected non-bipartite loop conditions are equivalent. If a variety (or an algebra) satisfies a non-trivial loop condition, we call it *loop-producing*. Since any finite loopless digraph can be homomorphically mapped to a large enough clique, we get the following corollary of that result.

Proposition 2.4. *Let* $\mathcal{V}$ *be a variety. The following are equivalent.*

1. $\mathbb{V}$ *is a loop-producing variety,*

2. *Let* $\mathbb{K}_\omega$ *denote the infinite countable clique, and let* $\mathbf{F}$ *be a* $\mathcal{V}$*-free algebra freely generated by the nodes of* $\mathbb{K}_\omega$*. Then the graph generated by edges of* $\mathbb{K}_\omega$ *has a loop.*

3. $\mathbb{V}$ *has a 6-ary Siggers term* $s(x, x, y, y, z, z) = s(y, z, z, x, x, y)$*,*

4. $\mathbb{V}$ *satisfies all undirected non-bipartite loop conditions,*

5. *Whenever* $\mathbf{A} \in \mathcal{V}$ *is compatible with a digraph* $\mathbb{G}$ *such that there is homomorphism* $\mathbb{C}_{2n+1} \to \mathbb{G}$ *for some* n*, then* $\mathbb{G}$ *has a loop.*

Less formally, it suffices to test the loop-producing property on large cliques, but then it can be applied to any symmetric cycle of odd length.

3 The proof of the loop condition collapse

In this section, we prove the following.

Theorem 3.1. *Let $\mathbb{G}$ be a strongly connected digraph with algebraic length 1, and $\mathcal{V}$ be a loop-producing variety. Then $\mathcal{V}$ satisfies the $\mathbb{G}$ loop condition.*

From basic properties of loop conditions we then obtain the following corollaries.

Corollary 3.1. *Let $\mathbf{A}$ be a loop-producing algebra compatible with a digraph $\mathbb{G}$. If $\mathbb{G}$ has a strongly connected component with algebraic length one, then $\mathbb{G}$ has a loop.*

Proof. It follows from Theorem 3.1 and Proposition 2.2, implication (1) $\Rightarrow$ (3). ☐

Corollary 3.2. *All the non-trivial loop conditions that have a strongly connected component of algebraic length one are equivalent.*

Proof. Any variety satisfying such a loop conditions is loop-producing by non-triviality. Conversely, any loop-producing variety satisfy all the loop conditions with algebraic length one by Theorem 3.1. Finally, all the non-trivial loop conditions that have a strongly connected component of algebraic length one are satisfied by Proposition 2.3. ☐

Corollary 3.3. *Any loop-producing variety has a 4-ary Siggers term $s(r,a,r,e) = s(a,r,e,a)$.*

Proof. This follows directly from Theorem 3.1 since the rare-area loop condition is an example of a strongly connected loop condition with algebraic length one. ☐

3.1 Digraph definitions

We provide three concrete types of digraphs that will serve as intermediate steps in the proof of Theorem 3.1.

Definition 3.2. *Given cycle lengths $a, b \geq 1$, we define the digraph $\mathrm{DCP}(a,b)$ (directed cycle pair) as follows. The nodes are $A_0, \ldots, A_{a-1}$ and $B_0, \ldots, B_{b-1}$ such that $A_0 = B_0$ while all the other are different. Edges goes from A_i to A_{i+1} modulo a and from B_i to B_{i+1} modulo b.*

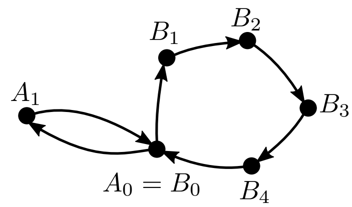

Figure 2: The digraph $\mathrm{DCP}(2, 5)$.

Definition 3.3. *For a given path length $k \geq 1$, number $l \geq 0$ of loop symbols, and cycle size $c \geq 1$, the digraph $\mathrm{CCLW}(k, l, s)$ (cycle walks) is defined as follows. Let $\mathcal{A} = \{0, 1, \ldots, c-1\}$ and $\mathcal{L}$ be an alphabet of size l not containing $\emptyset$. The nodes of $\mathrm{CCLW}(k, l, c)$ are the sequences $(a_1, l_1, a_2, l_2, \ldots . l_{k-1}, a_k)$ such that:*

- *every $a_i \in \mathcal{A}$ and every $l_i \in \mathcal{L} \cup \{\emptyset\}$,*
- *loop symbols do not repeat, that is, if $l_i = l_j$, then $i = j$ or $l_i = l_j = \emptyset$,*
- *If $l_i \in \mathcal{L}$, then $a_i = a_{i+1}$. Otherwise $a_i \equiv a_{i+1} \pm 1 \pmod{c}$.*

There is an edge $(n_1, n_2) \in \mathrm{CCLW}(k, l, s)$ if there is a node n in $\mathrm{CCLW}(k+1, l, s)$ such that n_1 is a prefix of n and n_2 is a suffix of n.

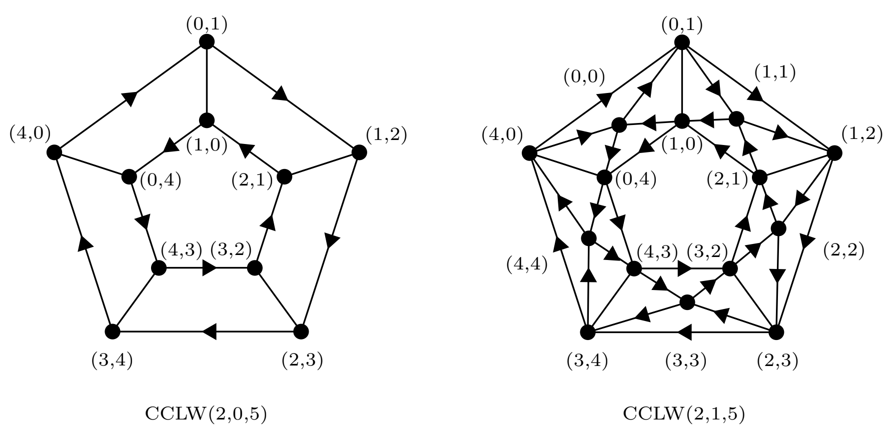

Figure 3: Examples of digraphs of cycle walks. The loop symbol is ommited since it is uniquely determined by adjacent items.

Definition 3.4. *For a given path length $k \geq 1$, number $l \geq 0$ of loop symbols, and alphabet size $s \geq 1$, the digraph $\mathrm{CLQP}(k, l, s)$ (clique paths) is defined as follows. Let $\mathcal{A}$ be an alphabet of size s and $\mathcal{L}$ be an alphabet of size l not containing $\emptyset$. The nodes of $\mathrm{CLQP}(k, l, s)$ are the sequences $(a_1, l_1, a_2, l_2, \ldots . l_{k-1}, a_k)$ such that:*

- *every* $a_i \in \mathcal{A}$ *and every* $l_i \in \mathcal{L} \cup \{\emptyset\}$,
- *loop symbols do not repeat, that is, if* $l_i = l_j$, *then* $i = j$ *or* $l_i = l_j = \emptyset$,
- *letters repeat if and only if they are connected by loop symbols, that is, for any* $i < j$: $a_i = a_j$ *if and only if all* $l_i, \ldots, l_{j-1} \in \mathcal{L}$.

There is an edge $(n_1, n_2) \in \mathrm{CLQP}(k, l, s)$ *if there is a node* n *in* $\mathrm{CLQP}(k+1, l, s)$ *such that* n_1 *is a prefix of* n *and* n_2 *is a suffix of* n.

Note that cycle walks and clique paths differ just in the third point where cycle walks allows repetition of letters but are more restrictive on the local behavior.

3.2 Proof outline

Let $\mathcal{V}$ be a fixed loop-producing variety. We begin by stating seven lemmas.

Lemma 1. *The* $\mathrm{CLQP}(1, 0, 3)$ *loop condition is satisfied in* $\mathcal{V}$.

Lemma 2. *For any* $k \geq 1, s \geq 1$, *the following implication between loop conditions holds for* $\mathcal{V}$:

$$\mathrm{CLQP}(k, 0, s) \Rightarrow \mathrm{CLQP}(k+1, s, 1).$$

Lemma 3. *For any* $k \geq 2, l \geq 1, s \geq 1$, *the following implication between loop conditions holds for* $\mathcal{V}$:

$$\mathrm{CLQP}(k, l, s) \Rightarrow \mathrm{CLQP}(k, l-1, 3s).$$

Lemma 4. *For any* $k \geq 1, l \geq 0, c \geq 1$, *the following implication between loop conditions holds for* $\mathcal{V}$:

$$\mathrm{CLQP}(k, l, 1) \Rightarrow \mathrm{CCLW}(k, l, c).$$

Lemma 5. *For any* $k \geq 2, l \geq 1, c \geq 3$, c *odd, the following implication between loop conditions holds for* $\mathcal{V}$:

$$\mathrm{CCLW}(k, l, c) \Rightarrow \mathrm{CCLW}(k, l-1, c).$$

Lemma 6. *For any odd* $c \geq 3$ *there is some* $k \geq 2$ *such that the following implication between loop conditions holds for* $\mathcal{V}$:

$$\mathrm{CCLW}(k, 0, c) \Rightarrow \mathrm{DCP}(2, c).$$

Lemma 7. *For any* $p, b \geq 1$, b *odd, the following implication between loop conditions holds for* $\mathcal{V}$:

$$\mathrm{DCP}(2^p, 2^{p+1} + b) \Rightarrow \mathrm{DCP}(2^{p+1}, b).$$

First, let us see how Theorem 3.1 follows from these lemmas. Using Lemma 2 once and Lemma 3 repeatedly yields the implication $\mathrm{CLQP}(k,0,s) \Rightarrow \mathrm{CLQP}(k+1,0,3^s)$ for any $k,s \geq 1$. Since the CLQP$(1,0,3)$ loop condition holds in $\mathcal{V}$ by Lemma 1, for any $k \geq 1$ there is $s \geq 3$ such that the CLQP$(k,0,s)$ loop condition also holds in $\mathcal{V}$.

Therefore by Lemmas 2, 4, also the CCLW(k,l,c) loop condition holds in $\mathcal{V}$ for any $k \geq 2, c \geq 1$ and some l depending on k. If c is odd and at least 3, we can reduce the number of loop symbols s to zero by repeated application of Lemma 5. So the CLQP$(k,0,c)$ loop condition holds in $\mathcal{V}$ for any $k \geq 2, c \geq 3$, c odd. Thus by Lemma 6 also the $DCP(2,c)$ loop condition holds for any odd $c \geq 3$.

Now, the DCP$(2^p,c)$ loop condition holds for any odd $c \geq 3$ and for any $p \geq 1$. This is done by induction on p where the induction step is given by Lemma 7. Finally, consider any strongly connected loopless digraph $\mathbb{G}$ with algebraic length 1. Digraph $\mathbb{G}$ is a homomorphic image of a directed 2^p-cycle for large enough p. Also $\mathbb{G}$ is a homomorphic image of a directed c-cycle where c is odd and large enough. Therefore $\mathbb{G}$ is a homomorphic image of DCP$(2^p,c)$ for large enough p,c, so the $\mathbb{G}$ loop condition is satisfied in $\mathcal{V}$.

It remains to check the validity of our 7 lemmas.

Lemma 1 is satisfied by Proposition 2.4 since CLQP$(1,0,3)$ is isomorphic to the undirected triangle. Lemma 2 also follows from an isomorphism of the appropriate digraphs. Sequence $(a_1,\emptyset,a_2,\emptyset,\ldots,a_k)$ in CLQP$(k,0,s)$ can be identified with the sequence $(x,a_1,x,a_2,\ldots,a_k,x)$ in CLQP$(k+1,s,1)$ where x is the only letter in the apropriate alphabet.

Lemma 4 is simple as well. the CCLW(k,l,c) loop condition is implied by CLQP$(k,l,1)$ just because CLQP$(k,l,1)$ is a subdigraph of CCLW(k,l,c).

Lemmas 3, 5, 6, 7 deserve bigger attention.

3.3 Proof of Lemma 3

Lemma 3. *For any $k \geq 2, l \geq 1, s \geq 1$, the following implication between loop conditions holds for $\mathcal{V}$:*

$$\mathrm{CLQP}(k,l,s) \Rightarrow \mathrm{CLQP}(k,l-1,3s).$$

Assume that the CLQP(k,l,s) loop condition is satisfied in $\mathcal{V}$. To show that the CLQP$(k,l-1,3s)$ loop condition is satisfied in $\mathcal{V}$, consider a digraph $\mathbb{G}$ compatible with an algebra in $\mathcal{V}$ such that CLQP$(k,l-1,3s)$ is a subdigraph of $\mathbb{G}$. It suffices to show that $\mathbb{G}$ has to contain a loop.

Let us construct another digraph $\mathbb{H}$, pp-constructed from $\mathbb{G}$. Nodes of $\mathbb{H}$ are defined as the homomorphisms CLQP$(k,l-1,s) \to \mathbb{G}$.

To define the edges of $\mathbb{H}$, we investigate the digraph CLQP(k,l,s) in more detail. Let $L_1,\ldots,L_l$ be the loop symbols of CLQP(k,l,s) and $\{L_1,\ldots,L_{l-1}\}$ be the loop symbols of CLQP$(k,l-1,s)$. Then we can view CLQP$(k,l-1,s)$ as an induced subgraph of CLQP(k,l,s). There are no extra edges since $k \geq 2$. Let $\mathbb{V}$ denote the digraph induced on all nodes of CLQP(k,l,s) containing the symbol L_l. So the set of all vertices is decomposed into $\mathrm{CLQP}(k, l-1, s)$ and $\mathbb{V}$.

Consider $v_0, v_1 \colon \mathrm{CLQP}(k, l-1, s) \to \mathbb{G}$, two vertices of $\mathbb{H}$. By definition, there is an edge $v_0 \xrightarrow{\mathbb{H}} v_1$ if and only if there is a digraph homomorphism $f \colon \mathbb{V} \to \mathbb{G}$ such that for any edge (x, y) in $\mathrm{CLQP}(k, l, s)$:

- If $x \in \mathrm{CLQP}(k, l-1, s)$ and $y \in \mathbb{V}$, then $v_0(x) \xrightarrow{\mathbb{G}} f(y)$.
- If $y \in \mathrm{CLQP}(k, l-1, s)$ and $x \in \mathbb{V}$, then $v_1(y) \xleftarrow{\mathbb{G}} f(x)$.

Claim 3.5. *There is an undirected triangle in* $\mathbb{H}$.

To show the claim, let $\mathcal{A}_s, \mathcal{A}_{3s}$ be alphabets of $\mathrm{CLQP}(k, l-1, s)$, $\mathrm{CLQP}(k, l-1, 3s)$ respectively. Fix three injective mappings $u_1, u_2, u_3 \colon \mathcal{A}_s \to \mathcal{A}_{3s}$ with pairwise disjoint images. These mappings extend naturally to three homomorphisms $\mathrm{CLQP}(k, l-1, s) \to \mathrm{CLQP}(k, l-1, 3s)$, which are vertices of $\mathbb{H}$ since $\mathrm{CLQP}(k, l-1, 3s)$ is a subdigraph of $\mathbb{G}$. We claim that these three nodes are pairwise adjacent. By symmetry, it suffices to show that there is an edge (u_1, u_2) in $\mathbb{H}$. We define a homomorphism $f \colon \mathbb{V} \to \mathrm{CLQP}(k, l-1, 3s)$ as follows. A general node of $\mathbb{V}$ of the form

$$(a_1, l_1, \ldots, a_i, l_i = L_l, a_{i+1}, l_{i+1}, \ldots, a_n)$$

is mapped to the node

$$(u_1(a_1), l_1, \ldots, u_1(a_i), \emptyset, u_2(a_{i+1}), l_{i+1}, \ldots, u_2(a_n))$$

Such f is a digraph homomorphism $\mathbb{V} \to \mathbb{G}$ satisfying the required properties, so there is an edge $u_1 \xrightarrow{\mathbb{H}} u_2$.

Since the variety $\mathcal{V}$ is a loop-producing variety and digraph $\mathbb{H}$ contains an undirected triangle, $\mathbb{H}$ contains a loop by Proposition 2.4. By definition of $\mathbb{H}$, such a loop witnesses the existence of a homomorphism $\mathrm{CLQP}(k, l, s) \to G$. So by the $\mathrm{CLQP}(k, l, s)$ loop condition, there is a loop in $\mathbb{G}$.

3.4 Proof of Lemma 5

Lemma 5. *For any $k \geq 2, l \geq 1, c \geq 3$, c odd, the following implication between loop conditions holds for $\mathcal{V}$:*

$$\mathrm{CCLW}(k, l, c) \Rightarrow \mathrm{CCLW}(k, l-1, c).$$

The approach is very similar to the proof of Lemma 3. However, we will repeat the reasoning because of the slight differences.

Assume that the $\mathrm{CCLW}(k, l, c)$ loop condition is satisfied in $\mathcal{V}$ and $\mathrm{CCLW}(k, l-1, c)$ is a subdigraph of a digraph $\mathbb{G}$ compatible with an algebra in $\mathcal{V}$. We want to find a loop in $\mathbb{G}$. We pp-construct a digraph $\mathbb{H}$. Its nodes are all the digraph homomorphisms $\mathrm{CCLW}(k, l-1, c) \to \mathbb{G}$.

The edges of $\mathbb{H}$ are defined analogously to the previous case. We decompose the nodes of $\mathrm{CCLW}(k,l,c)$ into two induced subraphs: $\mathrm{CCLW}(k,l-1,c)$ and $\mathbb{V}$. There is an edge $A \overset{\mathbb{H}}{\to} B$ if there is a digraph homomorphism $f\colon \mathbb{V} \to \mathbb{G}$ such that the pair of homomorphisms (A,f) preserves the edges from $\mathrm{CCLW}(k,l-1,c)$ to $\mathbb{V}$ and the pair of homomorphisms (B,f) preserves the edges in the other direction.

Claim 3.6. *Digraph $\mathbb{H}$ contains an undirected cycle of length c.*

Indeed, let $\mathcal{A}$ denote the set $\{0,1,\dots,c-1\}$. Let $u_i\colon \mathcal{A} \to \mathcal{A}$ denote the mapping

$$u_i(x) \equiv i + x \pmod c.$$

Since these mappings are also endomorphisms of $\mathbb{C}_n$, these mappings can be extended to homomorphisms $\mathrm{CCLW}(k,l-1,c) \to \mathrm{CCLW}(k,l-1,c)$. So we can view them as nodes of $\mathbb{H}$.

There are edges $u_x \overset{\mathbb{H}}{\to} u_y$ for $y = x \pm 1 \pmod c$. To verify it, we construct a homomorphism $\mathbb{V} \to \mathrm{CCLW}(k,l-1,c)$ testifying that. An element of $\mathbb{V}$

$$(a_1, l_1, \dots, a_i, l_i = L_l, a_{i+1}, l_{i+1}, \dots, a_n),$$

where L_l denotes the one extra loop symbol, is mapped to

$$(u_x(a_1), l_1, \dots, u_x(a_i), l_i = \emptyset, u_y(a_{i+1}), l_{i+1}, \dots, u_y(a_n))$$

in $\mathrm{CCLW}(k,l-1,c)$. Note that the $a_i = a_{i+1}$ because of the loop symbol L_i, so $u_y(a_{i+1}) \equiv u_x(a_i) \pm 1$. Therefore the position i meets the requirement for $\mathrm{CCLW}(k,l-1,c)$.

Since there is a symmetric cycle $\mathbb{C}_c$ in $\mathbb{H}$ and $\mathcal{V}$ is a loop-producing variety, there is a loop in $\mathbb{H}$ by Proposition 2.4 (5). A loop in $\mathbb{H}$ corresponds to a digraph homomorphism $\mathrm{CCLW}(k,l,c) \to \mathbb{G}$. Finally, the $\mathrm{CCLW}(k,l,c)$ loop condition gives a loop in $\mathbb{G}$.

3.5 Proof of Lemma 6

Lemma 6. *For any odd $c \geq 3$ there is some $k \geq 2$ such that the following implication between loop conditions holds for $\mathcal{V}$:*

$$\mathrm{CCLW}(k,0,c) \Rightarrow \mathrm{DCP}(2,c).$$

Actually, this is true because there is a digraph homomorphism $\mathrm{CCLW}(k,0,c) \to \mathrm{DCP}(2,c)$. To show it, let us reformulate the existence of a homomorphism a bit.

Consider a digraph homomorphism $a\colon \mathbb{Z} \to \mathbb{C}_c$ which contains all the finite walks on $\mathbb{C}_c$. This is possible since there are just countable number of the finite walks and $\mathbb{C}_c$ is connected. Let us view the mapping $a\colon \mathbb{Z} \to \mathbb{C}_c$ (and later defined $x\colon \mathbb{Z} \to \mathrm{DCP}_c$ as well) as a sequence instead of a function and write a_i instead of $a(i)$.

Claim 3.7. *There is a digraph homomorphism $x\colon \mathbb{Z} \to \mathrm{DCP}(2,c)$ and a number $k \geq 1$ such that whenever*

$$(a_{i-k}, a_{i-k+1}, \ldots, a_{i+k}) = (a_{j-k}), a_{j-k+1}, \ldots, a_{j+k})$$

for some integers i, j, then also $x_i = x_j$.

First, let us see how the claim proves Lemma 6. The value x_i is determined only by the $(2k+1)$-tuple $(a_{i-k}, \ldots, a_{i+k})$. By construction of a, any such walk on $\mathbb{C}_c$ of length $(2k+1)$ occurs among such tuples. So there is a well defined mapping $\mathrm{CCLW}(2k+1, 0, c) \to \mathrm{DCP}(2,c)$ given by

$$(a_{i-k}, \emptyset, a_{i-k+1}, \emptyset, \ldots, a_{i+k}) \mapsto x_i.$$

Finally, this mapping is a homomorphism since the sequence a covers all the walks of length $(2k+2)$ corresponding to the edges of $\mathrm{CCLW}(2k+1, 0, c)$.

Now, we prove the claim. Let S be the set of all the integers i satisfying

1. $a_i = 0$ or
2. $a_i \equiv 0 \pmod 2$ and all the values $a_{i-c+1}, \ldots, a_{i+c-1}$ are nonzero.

Observe that the difference between consecutive elements of S can not exceed $2c$. Indeed, whenever there are two zeros a_i, a_j in the sequence a, $i + 2c < j$, and there are no zeros between them, there is at least one integer in the middle assigned to S by the rule 2.

On the other hand, whenever there are two integers in S such that their difference is less than c, then the difference is even.

Given these facts, we can define the sequence x. For every $i \in S$ we set $x_i = A_0 = B_0$, then we fill the even spaces by alternation

$$A_0, A_1, A_0, A_1, \ldots, A_0$$

and the odd spaces greater than c by

$$B_0, B_1, \ldots B_{c-1}, A_0, A_1, A_0, A_1, \ldots, A_0.$$

Such a definition of x_i depends only on $S \cap \{i - 2c + 1, \ldots, i + 2c - 1\}$ and the definition of S depends just on neighborhoods with radius c. So $k = 3c$ completes the claim.

3.6 Proof of Lemma 7

Lemma 7. *For any $p, b \geq 1$, b odd, the following implication between loop conditions holds for $\mathcal{V}$:*

$$\mathrm{DCP}(2^p, 2^{p+1} + b) \Rightarrow \mathrm{DCP}(2^{p+1}, b).$$

Let the $\mathrm{DCP}(2^p, 2^{p+1}+b)$ loop condition be satisfied in $\mathcal{V}$ and let $\mathbb{G}$ be a digraph compatible with an algebra $\mathbf{A} \in \mathcal{V}$ having $\mathrm{DCP}(2^{p+1}, b)$ as its subdigraph. We need to show that $\mathbb{G}$ contains a loop.

Every node in $\mathrm{DCP}(2^{p+1}, b)$ is contained in a directed cycle of length $2^{p+1}+b$. Since this property is pp-definable, we can assume without lost of generality that all nodes of $\mathbb{G}$ are contained in such a cycle.

On the same set of vertices, we define another digraph $\mathbb{H}$ using second relational power of the edges of $\mathbb{G}$, that is (x, y) is an $\mathbb{H}$-edge if and only if there is a directed $\mathbb{G}$-walk of length 2 from x to y. Since $\mathbb{H}$ is pp-defined from $\mathbb{G}$, it is compatible with $\mathbf{A}$.

It is still true that every node of $\mathbb{H}$ is contained in a directed $\mathbb{H}$-cycle of length $2^{p+1}+b$. Moreover, the nodes $A_0, A_2, A_4, \ldots, A_{2^{p+1}-2}$ form a cycle of length 2^p. Therefore, there is a homomorphism $\mathrm{DCP}(2^p, 2^{p+1}+b) \to \mathbb{H}$. Using the assumption on $\mathcal{V}$, we get a loop in $\mathbb{H}$.

Such a loop corresponds to an undirected edge in $\mathbb{G}$. Any directed cycle of an even length can be homomorphically mapped to such an edge, in particular the cycle of length 2^p. Since every node of $\mathbb{G}$ belongs to a cycle of length $2^{p+1}+b$, there is a digraph homomorphism $\mathrm{DCP}(2^p, 2^{p+1}+b) \to \mathbb{G}$. Second application of the loop condition finished the proof.

4 Example loop-producing classes

In this section, we compare our result with standard classes in universal algebra, and show which of them are loop-producing.

We start by providing an alternative proof of Theorem 1.2 that finite Taylor algebras have a Siggers term. Our approach has a slightly weaker outcome than Theorem 1.3 (finite directed loop lemma) since we require strong conectedness. On the other hand, we managed to reduce the necessity of finiteness to a single step in the proof. That gives a hope for generalizations under weaker assumptions than local finiteness. One such generalization could be into the class of so called oligomorphic structures, which are already known to satisfy their own version of loop lemma, so called pseudoloop lemma. Details can be found in [5].

A variety $\mathcal{V}$ is said to be *locally finite*, if any finitely generated $\mathbf{A} \in \mathcal{V}$ is finite. Conversely, any variety generated by a single finite algebra is locally finite, which explains the equivalence between Theorem 1.2 and the following formulation.

Theorem 4.1. *Let $\mathcal{V}$ be a locally finite variety having a (not necessarilly idempotent) Taylor term, that is, a k-ary term t satisfying some k identities of the form*

$$t(x, ?, ?, ..., ?, ?) = t(y, ?, ?, ..., ?, ?) = s_1(x, y),$$
$$t(?, x, ?, ..., ?, ?) = t(?, y, ?, ..., ?, ?) = s_2(x, y),$$
$$\vdots$$

$$t(?, ?, ?, ..., ?, x) = t(?, ?, ?, ..., ?, y) = s_k(x, y).$$

where every question mark stands for either x or y. Then $\mathcal{V}$ is loop-producing.

Proof. We check that by verifying that $\mathcal{V}$ satisfies the $\mathbb{K}_{3k}$ loop condition. Let $\mathbf{F}$ be the free algebra generated by $3k$ elements, and $\mathbb{G}$ be a graph compatible with it containing $3k$-clique as a homomorphic image. We are supposed to prove that $\mathbb{G}$ contains a loop.

To do that, it is sufficient to prove that whenever $\mathbb{G}$ contains a clique of size $c \geq 3k$ as a homomorphic image, it also contains a clique of size $c+1$. Therefore we will be able to gradually increase the size of the clique, and eventually exceed the size of algebra $\mathbf{F}$ which is finite since $\mathcal{V}$ is a locally finite variety. The existence of a loop is then a direct consequence of the pigeonhole principle.

So let us suppose that there is a clique with elements A_i, B_i, C_i, D_j for $1 \leq i \leq k$ and $1 \leq j \leq c - 3k$. We are going to fill a matrix $a_{i,j}$ of size $c+1 \times k$ in such a way that the Taylor term t applied row-wise outputs the desired clique of size $c+1$. We write $a_{ii} = A_i, a_{ij} = B_i$ for $i \neq j$, $1 \leq i, j \leq k$. So the values $t(a_{i,1}, \ldots, a_{i,k})$ form a clique for $i \leq k$ since we used disjoint parts of the original clique in separate rows. Now, we fill in the next k rows of the matrix. For a row $k+i$, where $1 \leq i \leq k$, we use the elements A_i, C_i in such a way that

$$t(a_{k+i,1}, a_{k+i,2}, \ldots, a_{k+i,k}) = s_i(A_i, C_i)$$

and moreover $a_{k+i,i} = A_i$. This is possible by the i-th Taylor identity. If we apply the Taylor term to the first $2k$ rows now, we still get a clique. In particular, there is an edge between

$$t(a_{i,1}, a_{i,2}, \ldots, a_{i,k}) = t(B_i, \ldots, B_i, A_i = a_{i,i}, B_i, \ldots, B_i)$$

and

$$t(a_{k+i,1}, a_{k+i,2}, \ldots, a_{k+i,k}) = s_i(A_i, C_i)$$

since there is a representation of

$$s_i(A_i, C_i) = t(b_{i,1}, b_{i,2}, \ldots, b_{i,k})$$

such that all $b_{i,j}$ are equal to A_i or C_i and moreover $b_{i,i} = C_i$.

Since $a_{i,i} = a_{k+i,i} = A_i$, every column contains at most $2n - 1$ distinct vertices. Hence, we just complete the $c + 1 - 2n$ remaining positions by the remaining $c - (2n - 1)$ distinct vertices, and the Taylor term applied to rows gives a clique of size $c + 1$. By this process, we gradually raise the size of the clique until we exceed the size of the algebra $\mathbf{F}$. Thus, we get a loop, and the variety is loop-producing. □

If we omit the assumption of local finiteness, we cannot hope for the Siggers term even under the assumption of idempotency, as shown in [10]. This counterexample can be even extended to a congruence meet-semidistributive variety that does not have a Siggers term. That directs us to the varieties satisfying a non-trivial congruence identity, equivalently; having a Hobby-McKenzie term. The corresponding finite property is "ommiting types (1), (5)".

Theorem 4.2. *Let $\mathcal{V}$ be a variety satisfying a non-trivial congruence identity (see [1]). Then $\mathcal{V}$ has a Siggers term.*

Proof. Among many characterization (see [1], Theorem A.2), we use the following one (number (9) in the reffered theorem). A variety $\mathcal{V}$ has a Hobby-KcKenzie term if and only if it has a sequence of terms $f_0, \dots, f_{2m+1}$ such that

(i) $\mathcal{V} \models f_0(x, y, u, v) \approx x$,

(ii) $\mathcal{V} \models f_i(x, y, y, y) \approx f_{i+1}(x, y, y, y)$ for even i,

(iii) $\mathcal{V} \models f_i(x, x, y, y) \approx f_{i+1}(x, x, y, y)$ and
$\mathcal{V} \models f_i(x, y, x, y) \approx f_{i+1}(x, y, x, y)$ for odd i,

(iv) $\mathcal{V} \models f_{2m+1}(x, y, u, v) \approx v$.

We can easily check that all the terms f_i have to be idempotent, that is $f_i(x, x, x, x) = x$ for any i and x. We can assume that all terms of $\mathcal{V}$ are idempotent, otherwise we consider the idempotent reduct.

Let $\mathbf{F}$ be a $\mathcal{V}$-free algebra generated by an infinite countable set X. By Proposition 2.4 (ii), it suffices to prove the following. If $\mathbb{G} = (A, G)$ is a graph, where the binary relation G is generated by distinct pairs of X, then R contains a loop. Since our clique is infinite, and every element $x \in \mathbf{F}$ is generated by finitely many elements of X, any such element is $\mathbb{G}$-adjacent to all but finitely many elements of X. Therefore, for any finite set $x_1, x_2, \dots, x_n \in \mathbf{F}$, we can find an element $y \in X$ such that $y - x_i$ for all $i = 1, \dots, n$.

Let $\hat{\phi}_1, \hat{\phi}_2 \colon X \to X$ be two injective mappings with disjoint images X_1, X_2, and let $\phi_1, \phi_2 \colon \mathbf{F} \to \mathbf{F}$ be their unique extensions to endomorphisms of $\mathbf{F}$.

Since $\mathbf{F}$ is idempotent and the relation G is full on $X_1 \times X_2$, we have an edge $\phi_1(x) - \phi_2(y)$ for every pair $x, y \in \mathbf{F}$. Since $\hat{\phi}_1, \hat{\phi}_2$ are injective, the mappings $\hat{\phi}_1, \hat{\phi}_2$ are endomorphisms of the relational structure $\mathbb{G}|_X$. Therefore, even the induced mappings ϕ_1, ϕ_2 are homomorphisms of the relational structure $\mathbb{G}$.

For $i = 0, 1, \dots, m$, we gradually construct sequences $\hat{a}_i, a_i, \hat{b}_i, b_i, \hat{c}_i, c_i, d_i$ such that

$$a_i - f_{2i}(x, a_i, a_i, a_i)$$

for all $x - \hat{a}_i$. We start with an arbitrary $\hat{a}_0 = a_0$, then $f_0(x, a_0, a_0, a_0) = x$ so the condition on a_0 is satisfied. Now consider a general $i \in \{0, 1, \dots, m\}$ and take a $\hat{b}_i - \hat{a}_i$. Then

$$a_i - f_{2i}(\hat{b}_i, a_i, a_i, a_i) = f_{2i+1}(\hat{b}_i, a_i, a_i, a_i) \stackrel{\text{def}}{=} b_i.$$

so $b_i - a_i$. We continue by taking an element $\hat{c}_i \in X$ such that $\hat{c}_i - \hat{b}_i, a_i$. Then

$$c_i \stackrel{\text{def}}{=} f_{2(i+1)}(\hat{c}_i, b_i, \hat{c}_i, b_i) = f_{2i+1}(\hat{c}_i, b_i, \hat{c}_i, b_i) - f_{2i+1}(\hat{b}_i, a_i, a_i, a_i) = b_i$$

and

$$d_i \stackrel{\text{def}}{=} f_{2(i+1)}(\hat{c}_i, \hat{c}_i, b_i, b_i) = f_{2i+1}(\hat{c}_i, \hat{c}_i, b_i, b_i) - f_{2i+1}(\hat{b}_i, a_i, a_i, a_i) = b_i$$

Finally, consider an element $\hat{a}_{i+1} \in X$ such that $\hat{a}_{i+1} - \phi_1(\hat{c}_i), \phi_2(\hat{c}_i)$, define

$$a_{i+1} = f_{2(i+1)}(\hat{a}_{i+1}, \phi_1(c_i), \phi_2(d_i), \phi_2(d_i)),$$

and check the edges $a_{i+1} - \phi_1(c_i), \phi_2(d_i)$:

$$f_{2(i+1)}(\hat{a}_{i+1}, \phi_1(c_i), \phi_2(d_i), \phi_2(d_i)) - f_{2(i+1)}(\phi_1(\hat{c}_i), \phi_1(b_i), \phi_1(\hat{c}_i), \phi_1(b_i)),$$
$$f_{2(i+1)}(\hat{a}_{i+1}, \phi_1(c_i), \phi_2(d_i), \phi_2(d_i)) - f_{2(i+1)}(\phi_2(\hat{c}_i), \phi_2(\hat{c}_i), \phi_2(b_i), \phi_2(b_i)),$$

So $a_{i+1} - \phi_1(c_i)$ and $a_{i+1} - \phi_2(d_i)$. Therefore for any $x \in X$ such that $x - \hat{a}_{i+1}$, we have an edge

$$a_{i+1} = f_{2(i+1)}(\hat{a}_{i+1}, \phi_1(c_i), \phi_2(d_i), \phi_2(d_i)) - f_{2(i+1)}(x, a_{i+1}, a_{i+1}, a_{i+1}),$$

so we ensured the condition for a_{i+1}. We ultimately get an edge

$$a_m - f_{2m}(x, a_m, a_m, a_m) = f_{2m+1}(x, a_m, a_m, a_m) = a_m$$

for some $x - \hat{a}_m$. This gives a loop on a_m and finishes the proof. □

5 Cyclic terms

For $n \geq 2$, an n-ary cyclic term c_n is such a term that satisfies

$$c(x_1, x_2, x_3 \dots, x_n) = c(x_2, x_3, \dots, x_n, x_1).$$

The existence of an n-ary cyclic term is clearly a loop condition of the digraph $\mathbb{D}_n$. The importance of cyclic terms among loop conditions lies in the following theorem.

Theorem 5.1. *Every non-trivial strongly conencted loop condition C is either equivalent to the existence of a Siggers term, or to the existence of a certain cyclic term.*

Before we prove the theorem, we analyze what implications hold between the loop conditions of cyclic terms. We can directly get the following.

Proposition 5.1. *For $d, n \geq 2$, where d is a divisor of n, the following implications between loop conditions hold.*

(i) $\mathbb{D}_n \Rightarrow \mathbb{D}_d$,

(ii) $\mathbb{D}_n \Rightarrow \mathbb{D}_{n^2}$.

Proof. The implication (i) follows from the existence of a digraph homomorphism $\mathbb{D}_n \to \mathbb{D}_d$. For proving the implication (ii), we use a standard trick: Let $\mathbb{G} = (A, G)$ be a digraph containing $\mathbb{D}_{n^2}$ and compatible with an n-ary cyclic term c_n. We define a digraph $\mathbb{H}$, where $x \xrightarrow{\mathbb{H}} y$ if and only if there is a directed $\mathbb{G}$-walk from x to y of length n. The digraph $\mathbb{H}$ is pp-defined from $\mathbb{G}$, so it is compatible with c_n as well. Since $\mathbb{G}$ contained $\mathbb{D}_{n^2}$, the digraph $\mathbb{H}$ contains $\mathbb{D}_n$, and by compatibility with c_n, it has a loop. A loop in $\mathbb{H}$ corresponds to $\mathbb{D}_n$ in $\mathbb{G}$, so by second application of c_n, we get a loop in $\mathbb{G}$. This finishes the proof. □

Just from these two facts, we get the following. Let $\mathrm{rad}(n)$ denote the radical of an integer $n \geq 2$, that is the product of all distinct prime divisors of n.

Corollary 5.1. *If* $\mathrm{rad}(n_2)$ *divides* $\mathrm{rad}(n_1)$*, then the* $\mathbb{D}_{n_1}$ *loop condition implies the* $\mathbb{D}_{n_2}$ *loop condition. In particular, all the loop conditions* $\mathbb{D}_n$ *with a fixed radical of* n *are equivalent.*

Now, we are ready prove Theorem 5.1. Consider a strongly connected loop condition C. Let $n = al(\mathbb{G}_C)$ If $n = 1$, then C is equivalent to the existence of a Siggers term by Corollary 3.2. Otherwise, there is a digraph homomorphism $\mathbb{G}_C \to \mathbb{D}_n$, and a digraph homomorphisms $\mathbb{D}_{kn} \to \mathbb{G}_C$ for any k that is large enough. So there is a digraph homomorphism $\mathbb{D}_{n^e} \to \mathbb{G}_C$ for some exponent e, and since the $\mathbb{D}_n$ and the $\mathbb{D}_{n^e}$ loop conditions are equivalent, they are both equivalent to C.

We finish this section with an example showing that the implications we have, are optimal.

Theorem 5.2. *Let* C_1, C_2 *be two non-trivial strongly connected loop conditions. Let numbers* n_1, n_2 *denote the algebraic lengths of* $\mathbb{G}_{C_1}$*,* $\mathbb{G}_{C_2}$ *respectively. Then* C_1 *implies* C_2 *if and only if* $\mathrm{rad}(n_2)$ *divides* $\mathrm{rad}(n_1)$*.*

Proof. If $\mathrm{rad}(n_2)$ divides $\mathrm{rad}(n_1)$, then either $n_2 = 1$, or $n_1, n_2 \geq 2$. In the first case, C_2 is the weakest non-trivial loop condition, and therefore implied by C_1. In the second case, the conditions are equivalent to the existence of cyclic terms of arities n_1, n_2 respectively, and we get $C_1 \Rightarrow C_2$ from Corollary 5.1.

Now suppose that $\mathrm{rad}(n_2)$ does not divide $\mathrm{rad}(n_1)$. So there is a prime number p that divides n_2 but not n_1. Let A be the universe of the one-dimensional vector space over the p-element field. We equip the set A with operations of the following form: For every k-tuple $\alpha_1, \ldots, \alpha_k$ such that $\alpha_1 + \cdots + \alpha_k = 1$, we consider the k-ary operation $A^k \to A$:

$$(x_1, \ldots, x_k) \mapsto \alpha_1 x_1 + \cdots + \alpha_k x_k,$$

that is, the affine combination given by coefficients $\alpha_1 \ldots, \alpha_k$. We denote the final algebra $(A, \text{affine combinations})$ as $\mathbf{A}$. Note that all the term operations of $\mathbf{A}$ are still just some affine combinations. The algebra $\mathbf{A}$ satisfies C_1. If $n_1 \geq 2$, it suffices to find a cyclic affine combination. There is the following one:

$$c_1(x_1, \ldots, x_{n_1}) = \frac{1}{n_1} \sum_{i=1}^{n_1} x_i.$$

Note that $\frac{1}{n_1}$ is well defined since p does not divide n_1. If $n_1 = 1$, it suffices to find any satisfied loop condition. There is the Maltsev term $m(x, y, z) = x-y+z$ satisfying $m(y, x, x) = m(z, z, y)$.

On the other hand, there is no n_2-ary cyclic term in $\mathbf{A}$. For a contradiction, consider such a term c_2:

$$c_2(x_1, \ldots, x_{n_2}) = \sum_{i=1}^{n_2} a_i x_i.$$

Let 0 denote the zero vector in $\mathbf{A}$, and 1 denote an arbitrary non-zero vector. Since c_2 is cyclic, all the values

$$c_2(1, 0, \ldots, 0) = c_2(0, 1, 0, \ldots, 0) = \cdots = c_2(0, \ldots, 0, 1)$$

are equal. Therefore $a_1 = a_2 = \cdots = a_{n_2}$. But then

$$\sum_{i=1}^{n_2} a_i = n_2 a_1 = 0$$

since p divides n_2. This contradicts the fact that c_2 should be affine, and the sum is supposed to be equal to 1. □

6 Without strong conectedness

The case of loop conditions that are not strongly connected is largely open. It seems that there is no further big collapse of loop conditions up to equivalence. We provide two basic counterexamples concerning loop conditions that are not strongly connected, and then we provide a few of positive results.

6.1 Counter-examples

We have shown that all the non-trivial loop conditions corresponding to a digraph with a strongly connected component of algebraic length one are equivalent. In our first example we demonstrate that there is no other loop condition equivalent to them.

Example 6.1. *There is an algebra $\mathbf{A} = (A, s)$ and a compatible digraph $\mathbb{G} = (A, G)$ such that s is an idempotent 6-ary Siggers operation, $\mathbb{G}$ does not have a loop but any countable digraph without a strongly connected component with algebraic length 1 can be homomorphicly mapped into $\mathbb{G}$. Therefore $\mathbf{A}$ satisfies all the non-trivial loop conditions corresponding to a digraph with a strongly connected component with algebraic length one, but it satisfies no other non-trivial loop condition.*

Proof. Let $\mathbb{G}_0 = (A_0, G_0)$ be the disjoint union of directed cycles of all lengths $l \geq 2$. First, we show that $\mathbb{G}_0$ is compatible with an idempotent algebra $\mathbf{A}_0 = (A_0, s_0)$, where s_0 is a 6-ary Siggers operation.

We define $\sim$ as the smallest reflexive symmetric binary relation on A_0^6 that satisfies

$$(x, x, y, y, z, z) \sim (y, z, z, x, x, y)$$

for any $x, y, z \in A_0$. Note that this relation is also transitive: The only non-trivial way of applying transitivity is by interpretting a single six-tuple as both (x, x, y, y, z, z) and (y, z, z, x, x, y), but then $x = y = z$, and we generate just the reflexivity on constant tuples.

Let $\phi\colon A_0 \to A_0$ be the mapping that maps x to y such that $x \overset{\mathbb{G}_0}{\to} y$, and let S be a factor set $A_0^6/\sim$. The function ϕ is clearly compatible with $\sim$, so ϕ acts on S as well. In order to construct an idempotent Siggers operation, we need to find a mapping $s_0\colon S \to A_0$ that maps constant six-tuples to the appropriate elements and commutes with ϕ. The constant tuples has just one-element equivalent classes in $\sim$, and they are closed under ϕ, so we can handle them independently of the rest, and simply map them to the appropriate elements.

Now, consider any other element $\mathbf{x} \in S$ represented by a six-tuple

$$\bar{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in A_0^6.$$

The element $\mathbf{x}$ is in a ϕ-cycle of a finite length l that is less than or equal to the product of the lengths of the cycles containing $x_1, \ldots, x_6$. If $l \geq 2$, we can map that cycle into $\mathbb{G}$. It remains to check that the possibility $l = 1$ cannot happen. For a contradiction, assume that $\phi(\mathbf{x}) = \mathbf{x}$, that is, $\phi(\bar{x}) \sim \bar{x}$. Since $\phi(x_1) \neq x_1$, also $\phi(\bar{x}) \neq \bar{x}$. Therefore there are $x, y, z \in A_0$ such that

$$\bar{x} = (x, x, y, y, z, z) \text{ and } \psi(\bar{x}) = (y, z, z, x, x, y),$$

where ψ stands for either ϕ of ϕ^{-1}. Then we have $y = \psi(x) = z$ and $z = \psi(y) = x$, and we get a contradiction with the assumption that $\bar{x}$ is non-constant.

Now, we construct the promised example. Take the algebra $\mathbf{A}_1 = (\mathbb{Q}, s)$, where $\mathbb{Q}$ denotes the set of all rational numbers, and s_1 is the arithmetical mean of six elements. Note that s_1 satisfies the Siggers equation as well. We set $\mathbf{A} = \mathbf{A}_0 \times \mathbf{A}_1 = (A, s)$. The digraph $\mathbb{G} = (A, G)$ is defined as follows:

$$(x_1, q_1) \overset{\mathbb{G}}{\to} (x_2, q_2) \overset{\text{def}}{\Leftrightarrow} q_1 < q_2 \text{ or } (q_1 = q_2 \text{ and } (x_1, x_2) \in G_0).$$

Now, consider six edges $((x_i, p_i), (y_i, q_i)) \in G$ for $i = 1, 2, 3, 4, 5, 6$. Let $((x, p), (y, q))$ be the result of applying s to them. We have to check that the resulting edge is in G as well. We distinguish two cases. If $p_i < q_i$ for some i, then $p < q$, so $(x, p) \overset{\mathbb{G}}{\to} (y, q)$. Otherwise for all i, $p_i = q_i$ and $x_i \overset{\mathbb{G}_0}{\to} y_i$. In that case $p = q$ and $x \overset{\mathbb{G}_0}{\to} y$, so $(x, p) \overset{\mathbb{G}}{\to} (y, q)$.

Finally, we observe that any countable digraph $\mathbb{H}$ having no strongly connected component with algebraic length 1 can be homomorphically mapped to $\mathbb{G}$. Modulo the strong conectedness, the digraph $\mathbb{H}$ has no loop, so it can be mapped into $(\mathbb{Q}, <)$, since $(\mathbb{Q}, <)$ contains all strict partial orders as subdigrahs. Finally, every strongly connected component of $\mathbb{H}$ can be mapped into G_0 since it has algebraic length bigger than one. These two mappings together form the digraph homomorphism $\mathbb{H} \to \mathbb{G}$. □

Our second example shows that if we do not require at least some cycles in the digraph, we cannot get a loop even under quite strong assumptions. In particular, our example is locally finite, has a ternary near unanimity operation, and the compatible digraph forms an ω-categorical structure.

Example 6.2. *Let* $\mathbf{A} = (\{0,1\}, m)$ *be the two-element algebra, where* m *is the majority operation*

$$m(x,x,y) = m(x,y,x) = m(y,x,x) = x.$$

There is a smooth loopless digraph $\mathbf{G}$ *compatible with a subalgebra* $\mathbf{B} \leq \mathbf{A}^\omega$*, such that* $\mathbb{G}$ *contains as subdigraphs all countable digraphs without directed cycles.*

Proof. Regard elements of $\mathbf{A}^\omega$ as infinite sequences indexed by rational numbers, that is functions $a\colon \mathbb{Q} \to \{0,1\}$. For $q \in \mathbb{Q}$ define $b_q \in \mathbf{A}^\omega$ as follows

$$b_q(x) = \begin{cases} 0 & \text{if } x \leq q, \\ 1 & \text{if } x > q. \end{cases}$$

Then if $q_1, q_2, q_3 \in \mathbb{Q}$, the value $m(b_{q_1}, b_{q_2}, b_{q_3})$ equals the function b_q, where q is the median element of the triple (q_1, q_2, q_3). Therefore, the set $B = \{b_q : q \in \mathbb{Q}\}$ is a subuniverse, and we consider the subalgebra $\mathbf{B} = (B, m)$.

The digraph $\mathbb{G} = (B, G)$ is constructed by putting $(b_{q_1}, b_{q_2}) \in G$ if and only if $q_1 < q_2$. This digraph contains all countable digraphs without directed cycles since any countable strict partial order is a subdigraph of $(\mathbb{Q}, <)$.

It is not difficult to check that $\mathbb{G}$ is compatible with the operation m, that is that the strict order is compatible with median operation. Let m' denote the median of three rational numbers. Consider three pairs $p_1 < q_1$, $p_2 < q_2$, $p_3 < q_3$, we check $m'(p_1, p_2, p_3) < m'(q_1, q_2, q_3)$. Without loss of generality, $p_1 \leq p_2 \leq p_3$, so $m'(p_1, p_2, p_3) = p_2$. Both q_2 and q_3 are strictly greater than p_2, therefore $m'(q_1, q_2, q_3) > p_2 = m'(p_1, p_2, p_3)$. ☐

6.2 Partial positive results

Here we show a few loop conditions that are not strongly connected, and are satisfied by certain algebras. We begin with the definitions.

A *Maltsev term* is an idempotent ternary term m satisfying the equation $m(y,x,x) = m(z,z,y)$. By a *non-idempotent Maltsev term* we mean a term m satisfying this equation but not being necessarily idempotent. The existence of a Maltsev term is a loop condition given by the graph $x \to y \to z \leftarrow x$.

A *near unanimity term* (briefly *NU term*) is an n-ary term t such that $t(x, \ldots, x, y, x, \ldots, x) = x$ for any position of y. Here, we will be primarily interested in ternary NU term, that is, a term t such that $t(x,x,y) = t(x,y,x) = t(y,x,x) = x$.

By a *cone*, we mean the digraph given by $c_1 \to \cdots \to c_n \to c_1$, where $n \geq 2$, $a \to c_i$ for any $1 \leq i \leq n$, and $b \to a$.

Proposition 6.1. *Non-idempotent Maltsev term implies any weakly connected loop condition with algebraic length 1.*

Proof. Consider a digraph $\mathbb{G} = (A, G)$ having a weakly connected component of algebraic length 1, and compatible with an algebra $\mathbf{A}$ with a Maltsev term m. Since $\mathbb{G}$ has algebraic length 1, there is an oriented cycle $a_0, a_1, \ldots, a_{n-1}, a_0$ such that $a_i \overset{\mathbb{G}}{\to} a_{i+1}$ or $a_i \overset{\mathbb{G}}{\leftarrow} a_{i+1}$ for every $i = 0 \ldots, n-1$ (we calculate indices modulo n in this proof), and the number or right arrows exceeds the number of left arrows by one. We prove our proposition by induction on n. If $n = 1$, we have a loop.

If $n > 1$, we distinguish two cases.

(a) There is a zig-zag pattern in the cycle $a_i \overset{\mathbb{G}}{\to} a_{i+1} \overset{\mathbb{G}}{\leftarrow} a_{i+2} \overset{\mathbb{G}}{\to} a_{i+3}$, or $a_i \overset{\mathbb{G}}{\leftarrow} a_{i+1} \overset{\mathbb{G}}{\to} a_{i+2} \overset{\mathbb{G}}{\leftarrow} a_{i+3}$.

(b) There is no zig-zag pattern in the cycle, that is, each arrow is next to an arrow of the same direction.

If there is a zig-zag pattern, we pp-define a digraph $\mathbb{H} = (A, H)$ by $y_1 \overset{\mathbb{H}}{\to} y_2$ if there are x, z such that $y_1 \overset{\mathbb{G}}{\to} z \overset{\mathbb{G}}{\leftarrow} x \overset{\mathbb{G}}{\to} y_2$. Notice that $H \supset G$, and $\mathbb{H}$ builds a "bridge" $a_i \overset{\mathbb{H}}{\to} a_{i+3}$, or $a_{i+3} \overset{\mathbb{H}}{\to} a_i$ to the zig-zag pattern. So if we omit a_{i+1}, a_{i+2}, the cycle has still algebraic length one in $\mathbb{H}$. By induction hypothesis, Maltsev term imply the loop condition of that shorter cycle, so $\mathbb{H}$ has a loop. A loop in $\mathbb{H}$ corresponds to $x \overset{\mathbb{G}}{\to} y \overset{\mathbb{G}}{\to} z, x \overset{\mathbb{G}}{\to} z$, which is the digraph of the non-idempotent maltsev term. Therefore $\mathbb{G}$ has a loop.

If there is no zig-zag pattern in the cycle, we use the following pp-definition. $x_1 \overset{\mathbb{H}}{\to} x_2$ if there are y, z such that $x_1 \overset{\mathbb{G}}{\to} y \overset{\mathbb{G}}{\to} z \overset{\mathbb{G}}{\leftarrow} x_2$. Now, we have $x_1 \overset{\mathbb{H}}{\to} x_2$ whenever $x_1 \overset{\mathbb{G}}{\to} x_2$ and x_2 has an outcoming edge. Nevertheless, we can omit all the peaks from our cycle. Whenever we have

$$a_i \overset{\mathbb{G}}{\to} a_{i+1} \overset{\mathbb{G}}{\to} a_{i+2} \overset{\mathbb{G}}{\leftarrow} a_{i+3} \overset{\mathbb{G}}{\leftarrow} a_{i+4},$$

we can discard a_{i+2} since $a_i \overset{\mathbb{H}}{\to} a_{i+3}$. Therefore we get a shorter cycle in $\mathbb{H}$ following a loop in $\mathbb{H}$. As in the previous case, a loop in $\mathbb{H}$ corresponds to a digraph $x \overset{\mathbb{G}}{\to} y \overset{\mathbb{G}}{\to} z, x \overset{\mathbb{G}}{\to} z$ that implies a loop using the Maltsev term. □

Proposition 6.2. *Ternary near unanimity term implies the "cone" loop condition.*

Proof. Let $\mathbf{A} = (A, t)$ be an algebra where t is a ternary near unanimity term, and assume that there is a graph (denoted by arrows) compatible with $\mathbf{A}$. Moreover assume that a cone $a, b, c_1, \ldots, c_n$ is a subgraph of the graph. We need to prove that the graph contains a loop.

We set $x_n = c_n$ and recursively construct elements $x_{n-1}, x_{n-2}, \ldots, x_1 \in \mathbf{A}$: by $x_i = m(x_{i+1}, c_i.a)$.

We get a loop on x_1 by proving the following properties by induction

(i) $x_i \to c_1$ for all $n \geq i \geq 1$,

(ii) $a \to x_i$ for all $n \geq i \geq 1$,

(iii) $x_i \to x_{i+1}$ for all $n-1 \geq i \geq 1$.

When they are proved, there is a loop by

$$x_1 = m(x_1, x_1, b) \to m(x_2, c_1, a) = x_1.$$

The proof is straighforward. We check the first step of induction:

(i) $x_n = c_n \to c_1$,

(ii) $a \to c_n = x_n$,

(iii) $x_{n-1} = m(x_n, c_{n-1}, a) \to m(c_1, c_n, c_n) = x_n$,

and the induction step:

(i) $x_{i-1} = m(x_i, c_{i-1}, a) \to m(c_1, c_i, c_1) = c_1$,

(ii) $a = m(a, a, b) \to m(x_i, c_{i-1}, a) = x_{i-1}$,

(iii)

$$\begin{aligned} x_{i-1} = m(x_i, c_{i-1}, a) &\to m(x_{i+1}, c_i, x_{i+1}) = x_{i+1}, \\ x_{i-1} = m(x_i, c_{i-1}, a) &\to m(x_{i+1}, c_i, c_i) = c_i, \\ x_{i-1} = m(x_{i-1}, x_{i-1}, b) &\to m(x_{i+1}, c_i, a) = x_i. \end{aligned}$$

This finishes the proof. □

However, we don't have an answer to the following question.

Question 6.1. *Does the existence of a 4-ary near unanimity term*

$$t(y, x, x, x) = t(x, y, x, x) = t(x, x, y, x) = t(x, x, x, y) = x$$

imply any loop condition that is not equivalent to the $\mathbb{K}_3$ loop condition?

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